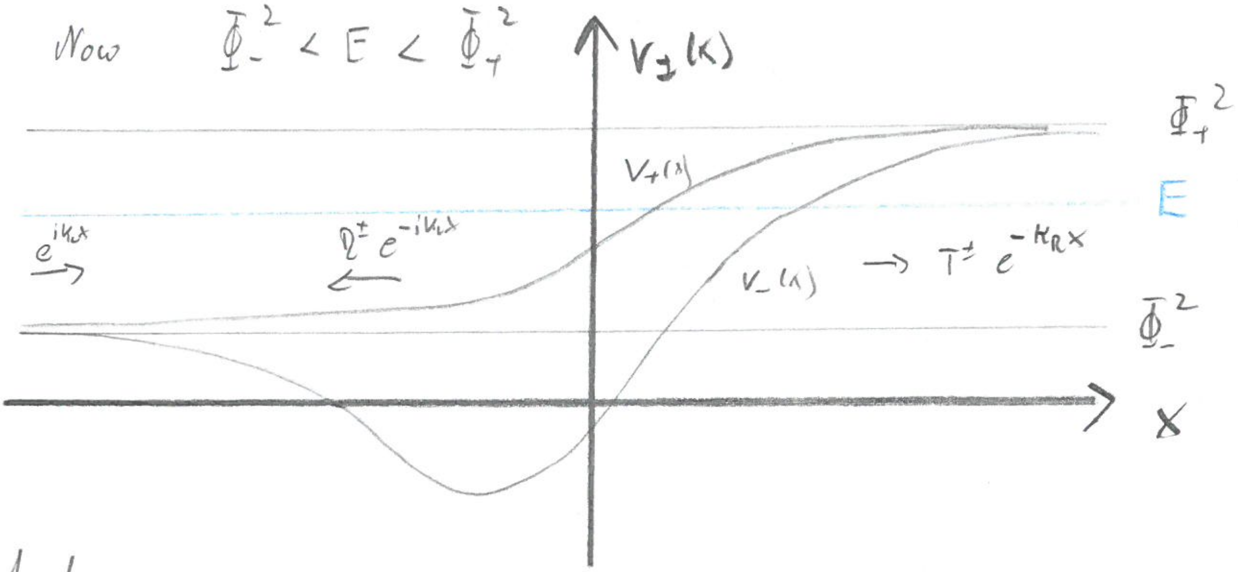
2. Homework

Problem 1: Following Exercise 6 we have

$$\lim_{x\to-\infty} V_\pm(x) = \lim_{x\to-\infty} \Phi^2(x) = \Phi_-^2$$

$$\lim_{x\to+\infty} V_\pm(x) = \lim_{x\to+\infty} \Phi^2(x) = \Phi_+^2$$

Now $\Phi_-^2 < E < \Phi_+^2$



Ansatz:

$$x\to-\infty: \quad \phi_E^\pm(x) \sim e^{ik_Lx} + R^\pm(E)\, e^{-ik_Lx} \quad \text{with} \quad k_L(E) := \frac{1}{\hbar}\sqrt{2m(E-\Phi_-^2)}$$

$$x\to+\infty: \quad \phi_E^\pm(x) \sim T^\pm(E)\, e^{-k_Rx} \quad \text{with} \quad k_R(E) := \frac{1}{\hbar}\sqrt{2m(\Phi_+^2-E)}$$

a) Consider for $x\to+\infty$:

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + V_\pm(x)\right)\phi_E^\pm(x) \approx \left(-\frac{\hbar^2}{2m}\partial_x^2 + \Phi_+^2\right) T^\pm(E)\, e^{-k_Rx}$$

$$= \left(-\frac{\hbar^2 k_R^2}{2m} + \Phi_+^2\right)\phi_E^\pm(x) \overset{!}{=} E\,\phi_E^\pm(x)$$

$$\leadsto \quad E = \Phi_+^2 - \frac{\hbar^2 k_R^2}{2m} \quad \leadsto \quad \underline{\underline{k_R = \frac{1}{\hbar}\sqrt{2m(\Phi_+^2-E)}}} \quad \#$$

SUSY Transformation:

$$\mathcal{N}\left(\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi(x)\right)\phi_E^-(x) = \phi_E^+(x)$$

• for $x \to -\infty$: $\mathcal{N}\left(\frac{\hbar}{\sqrt{2m}} ik_L + \Phi_-\right)e^{ik_Lx} + \mathcal{N}R^-\left(-\frac{\hbar ik_L}{\sqrt{2m}} + \Phi_-\right)e^{-ik_L|x|}$

$$= e^{ik_Lx} + R^+ e^{-ik_Lx}$$

$\curvearrowright$ $\mathcal{N} = \left(\frac{\hbar}{\sqrt{2m}} ik_L + \Phi_-\right)$ and $\mathcal{N}R^- = \dfrac{R^+}{\Phi_- - i\frac{\hbar k_L}{\sqrt{2m}}}$

$$\Rightarrow \boxed{R^+(E) = \frac{\Phi_- - i\sqrt{E-\Phi_+^2}}{\Phi_- + i\sqrt{E-\Phi_+^2}}\, R^-(E)}$$

Same as in Exercise 6

• for $x \to +\infty$: $\mathcal{N}T^-\left(-\frac{\hbar}{\sqrt{2m}}\kappa_R + \Phi_+\right)e^{-\kappa_R x} = T^+ e^{-\kappa_R x}$

$$\Rightarrow \boxed{T^+(E) = \frac{\Phi_+ - \sqrt{\Phi_+^2 - E}}{\Phi_- + i\sqrt{E-\Phi_-^2}}\, T^-(E)}$$

#

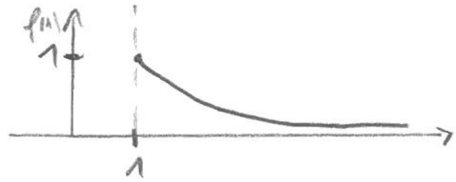
b) $|T^+(E)|^2 = \frac{1}{E}\left(\Phi_+ + \sqrt{\Phi_+^2 - E}\right)^2 |T^-(E)|^2 = f(\Phi_+/\sqrt{E})\,|T^-(E)|^2$

with $f(x) := x - \sqrt{x^2-1}$ for $x > 1$ as $\Phi_+^2 > E$

$f(1) = 1$

$f'(x) = 1 - \frac{x}{\sqrt{x^2-1}} \overset{!}{=} 0 \curvearrowright x = \sqrt{x^2-1}$ or $x^2 = x^2-1$ ↯ $\curvearrowright$ no extrema

$\lim\limits_{x\to\infty} f(x) = 0$



Hence $f(x)$ is strictly monotonic decreasing

$\Rightarrow f(x) < 1 \quad \forall\, x > 1$

$\Rightarrow |T^+(E)|^2 < |T^-(E)|^2 \quad \forall\, E < \Phi_+^2$ #

Problem 2: $\Phi(x) = \frac{\hbar}{\sqrt{2m}} \tanh x$

$$\Rightarrow \quad V_+(x) = \frac{\hbar^2}{2m} \qquad \text{and} \quad V_-(x) = \frac{\hbar^2}{2m}\left(1 - \frac{2}{\cosh^2 x}\right)$$

Ansatz: $\phi_E^+(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$ plane wave solution of free particle $k \in \mathbb{R}$

$$\leadsto \quad H_+ \phi_E^+(x) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{\hbar^2}{2m}\right)\frac{1}{\sqrt{2\pi}} e^{ikx} = \frac{\hbar^2}{2m}(k^2+1)\,\phi_E^+(x)$$

$$\Rightarrow \quad E = \frac{\hbar^2}{2m}(k^2+1) \quad \text{with } k \in \mathbb{R}$$

SUSY Trafo: $\left(-\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi(x)\right)\phi_E^+(x) = \frac{1}{N'}\,\phi_E^-(x)$

$$\leadsto \quad \phi_E^-(x) = N'\left(-\frac{\hbar}{\sqrt{2m}} ik + \Phi(x)\right)\phi_E^+(x) = N'\frac{\hbar}{\sqrt{2m}}\left(\tanh x - ik\right)\phi_E^+(x)$$

$$\Rightarrow \quad \boxed{\phi_E^-(x) = N\left(\tanh x - ik\right)\phi_E^+(x)} \quad \#$$

Cross Check: $\partial_x \phi_E^-(x) = N \frac{1}{\cosh^2 x} e^{ikx} + ik\,\phi_E^-(x)$

$$\partial_x^2 = -2\frac{\sinh x}{\cosh^3 x} N e^{ikx} + ik\frac{N}{\cosh^2 x} e^{ikx} + ik\,\partial_x \phi_E^-(x)$$

$$= -2\frac{\sinh x}{\cosh^3 x} N e^{ikx} + 2ik\frac{N}{\cosh^2 x} e^{ikx} - k^2 \phi_E^-(x)$$

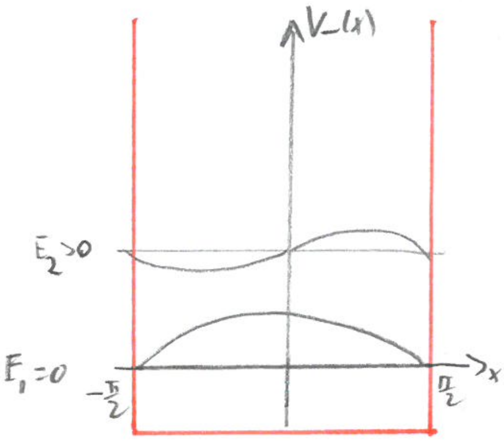
$$= \frac{2N e^{ikx}}{\cosh^2 x}\left(ik - \tanh x\right) - k^2\phi_E^-(x) = \left(-\frac{2}{\cosh^2 x} - k^2\right)\phi_E^-(x)$$

$$\leadsto \left[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{\hbar^2}{2m}\left(1 - \frac{2}{\cosh^2 x}\right)\right]\phi_E^-(x) = \frac{\hbar^2}{2m}\left(\frac{2}{\cosh^2 x} + k^2 + 1 - \frac{2}{\cosh^2 x}\right)\phi_E^-(x)$$

$$= \frac{\hbar^2}{2m}(k^2+1)\,\phi_E^-(x) = E\,\phi_E^-(x) \quad \#$$

Problem 3: $\quad \Phi(x) = \frac{\hbar}{\sqrt{2m}} \tan x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\Rightarrow \quad V_-(x) = -\frac{\hbar^2}{2m} \quad \text{and} \quad V_+(x) = \frac{\hbar^2}{2m}\left(\frac{2}{\cos^2 x} - 1\right)$$



a) Ansatz:

$$\phi_E^-(x) = \sqrt{\frac{2}{\pi}} \sin\left(nx + n\frac{\pi}{2}\right) \quad , n = 1, 2, 3, \ldots$$

$$\sim \quad \phi_E^-\left(\pm\frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi}} \sin\left(\pm\frac{\pi}{2}n + n\frac{\pi}{2}\right) = 0 \quad ✓$$

$$\partial_x^2 \phi_E^-(x) = -n^2 \phi_E^-(x)$$

$$\sim \quad -\frac{\hbar^2}{2m}\partial_x^2 \phi_E^-(x) = \frac{\hbar^2 n^2}{2m}\phi_E^-(x) \quad \Rightarrow \quad \underline{\underline{E_n = \frac{\hbar^2}{2m}(n^2 - 1)}}$$

$$\text{Norm:} \int_{-\frac{\pi}{2}}^{\pi/2} dx \left|\phi_E^-(x)\right|^2 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\pi/2} dx \sin^2\left(nx + n\frac{\pi}{2}\right) \underset{z = n(x + \frac{\pi}{2})}{=} \frac{2}{\pi n}\int_0^{n\pi} dz \sin^2 z = \frac{2}{\pi}\int_0^{\pi} dz \sin^2 z = \underline{\underline{1}}$$

b)

$$A\phi_0^-(x) = \left(\frac{\hbar}{\sqrt{2m}}\partial_x + \Phi(x)\right)\phi_0^-(x) = \frac{\hbar}{\sqrt{2m}}\left(\partial_x + \tan x\right)\sqrt{\frac{2}{\pi}}\sin\left(x + \frac{\pi}{2}\right)$$

$$= \frac{\hbar}{\sqrt{\pi m}}\left(\underbrace{\cos\left(x + \frac{\pi}{2}\right)}_{= -\sin x} + \tan x \underbrace{\sin\left(x + \frac{\pi}{2}\right)}_{= \cos x}\right) = \frac{\hbar}{\sqrt{\pi m}}\left(-\sin x + \tan x \cos x\right) = \underline{\underline{0}}$$

c)

$$\phi_{E_n}^+(x) = \frac{1}{\sqrt{E_n}} A \phi_{E_n}^-(x) \qquad n > 1$$

$$= \frac{1}{\sqrt{n^2 - 1}}\left(\partial_x + \tan x\right)\phi_{E_n}^-(x)$$

$$\underline{\underline{= \sqrt{\frac{2}{\pi(n^2 - 1)}}\left(n\cos\left(nx + \frac{\pi n}{2}\right) + \tan x \sin\left(nx + \frac{\pi n}{2}\right)\right)}}$$