# 1. HOMEWORK

**Problem 1:** $N=1$ SUSY $\leadsto$ $\{Q_1, Q_1\} = H$
or $2Q_1^2 = H$

a) Here $Q_1 := \frac{P}{\sqrt{4m}}$ $\leadsto$ $2Q_1^2 = \frac{P^2}{2m}$ #

Ground state is particle at rest, i.e $P|0\rangle = 0$

$\leadsto H|0\rangle = 0$ $\leadsto$ good SUSY

b) Let $P|k\rangle = \hbar k|k\rangle$ $k \in \mathbb{R}$

$\leadsto$ $Q_1|k\rangle = \frac{\hbar k}{\sqrt{4m}}|k\rangle$ and $|Q_1|\,|k\rangle = \frac{\hbar|k|}{\sqrt{4m}}|k\rangle$

$\leadsto$ $\Lambda|k\rangle = \frac{k}{|k|}|k\rangle = \pm|k\rangle$ for $k \gtrless 0$

$\leadsto$ Grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with $\mathcal{H}^+ := \mathrm{span}\{|k\rangle \,|\, k>0\}$
$\mathcal{H}^- := \mathrm{span}\{|k\rangle \,|\, k\leq 0\}$
have added ground state to $\mathcal{H}^-$

Algebra: $[\Lambda, H] = 0$ o.k.
$\Lambda^2 = 1$ o.k.
But $\{\Lambda, Q_1\} = 2\Lambda Q_1 = 2\frac{Q_1^2}{|Q_1|} = 2|Q_1| \neq 0$
does not fullfill $\{W, Q_1\} = 0$ ! $\leadsto$ Not a Witten operator

Note: The formal definition

$W = \frac{2}{H}QQ^+ - 1$ fails as for $N=1$ $Q = Q^+ = Q_1$

$\leadsto W = 0$

Problem 2: $N=2$ SUSY $\{Q_1, Q_2\}=0$ and $2Q_1^2 = H = 2Q_2^2$

Let $Q_1|q\rangle = q|q\rangle$ $q$ is real as $Q_1^\dagger = Q_1$

$\leadsto H|q\rangle = 2Q_1^2|q\rangle = 2q^2|q\rangle$ $\leadsto E_q := 2q^2 \geq 0$

and $H|q\rangle = E_q|q\rangle$

SUSY is unbroken if $q=0$ is eigenvalue of $Q_1$

Define: $|-q\rangle := \mathcal{N} Q_2 |q\rangle$

- Normalisation: $\langle -q|-q\rangle = \mathcal{N}^2 \langle q|Q_2^2|q\rangle = \frac{1}{2}\mathcal{N}^2\langle q|H|q\rangle = \mathcal{N}^2 q^2 \overset{!}{=} 1$

  $\leadsto$ $\mathcal{N} = \frac{1}{q}$, we allow for negative $\mathcal{N}$? (trivial phase)

  $\boxed{|-q\rangle = \frac{1}{q} Q_2|q\rangle}$

- Eigenstate of $Q_1$: $Q_1|-q\rangle = \frac{1}{q} Q_1 Q_2 |q\rangle = -\frac{1}{q} Q_2 Q_1|q\rangle = -Q_2|q\rangle$ $\leadsto$

  $\boxed{Q_1|-q\rangle = -q|-q\rangle}$

- Eigenstate of $H$: $H|-q\rangle = 2Q_1^2|-q\rangle = 2q^2|-q\rangle = E_q|-q\rangle$

  with <u>same</u> eigenvalue $E_q$

- Orthogonality: $\langle q|-q\rangle = \frac{1}{q}\langle q|Q_2|q\rangle = \frac{1}{q^2}\langle q|Q_2 Q_1|q\rangle$

  $= -\frac{1}{q^2}\langle q|Q_1 Q_2|q\rangle = \frac{-1}{q}\langle q|Q_2|q\rangle$

  $= -\langle q|-q\rangle$

  $\leadsto$ $\boxed{\langle q|-q\rangle = 0}$

## Problem 3:

Explicit matrix repr.:

$$Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \frac{P}{\sqrt{2m}} - i\Phi \\ \frac{P}{\sqrt{2m}} + i\Phi & 0 \end{pmatrix}$$

$$Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\frac{iP}{\sqrt{2m}} - \Phi \\ \frac{iP}{\sqrt{2m}} - \Phi & 0 \end{pmatrix}$$

$$\rightsquigarrow Q_1 Q_2 = \frac{1}{2} \begin{pmatrix} i\frac{P^2}{2m} + i\Phi^2 + \frac{1}{\sqrt{2m}}[\Phi, P] & 0 \\ 0 & -i\frac{P^2}{2m} - i\Phi^2 - \frac{1}{\sqrt{2m}}[P, \Phi] \end{pmatrix}$$

$$Q_2 Q_1 = \frac{1}{2} \begin{pmatrix} -i\frac{P^2}{2m} - i\Phi^2 + \frac{1}{\sqrt{2m}}[P, \Phi] & 0 \\ 0 & i\frac{P^2}{2m} + i\Phi^2 - \frac{1}{\sqrt{2m}}[\Phi, P] \end{pmatrix}$$

$$\Rightarrow \{Q_1, Q_2\} = 0 \quad \#$$

$$2Q_1^2 = \begin{pmatrix} \frac{P^2}{2m} + \Phi^2 + \frac{i}{\sqrt{2m}}[P, \Phi] & 0 \\ 0 & \frac{P^2}{2m} + \Phi^2 - \frac{i}{\sqrt{2m}}[P, \Phi] \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = H$$

$$2Q_2^2 = \quad \text{dto} \quad = H$$

$$\Rightarrow \boxed{\{Q_i, Q_j\} = H\,\delta_{ij}} \qquad N=2 \text{ SUSY algebra}$$

Note: $\frac{i}{\sqrt{2m}}[P, \Phi] = \frac{\hbar}{\sqrt{2m}}\Phi'(x)$

Problem 4: Recall

$$Q_A = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad Q_B = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \quad H_A = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \quad H_B = \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}$$

with $AB = 0 = BA \rightsquigarrow A^\dagger B^\dagger = 0 = B^\dagger A^\dagger$

- $[H_A, H_B] = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}\begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix} - \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}\begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} A\overbrace{A^\dagger B^\dagger}^{0}B & 0 \\ 0 & A^\dagger \underbrace{AB}_{0} B^\dagger \end{pmatrix} - \begin{pmatrix} B^\dagger \overbrace{BA}^{0}A^\dagger & 0 \\ 0 & B\underbrace{B^\dagger A^\dagger}_{0} A \end{pmatrix} = \underline{\underline{0}}$

- $[H_A, Q_B] = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ A^\dagger \underbrace{AB}_{0} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ BAA^\dagger & 0 \end{pmatrix} = \underline{\underline{0}} = [H, Q_B^\dagger]$

- $[H_B, Q_A] = \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix} = \begin{pmatrix} 0 & B^\dagger \overbrace{BA}^{0} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \overbrace{AB}^{0}B^\dagger \\ 0 & 0 \end{pmatrix} = \underline{\underline{0}} = [H_B, Q_A^\dagger]$

- $[Q_B, Q_A] = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} -AB & 0 \\ 0 & BA \end{pmatrix} = \underline{\underline{0}} = [Q_B^\dagger, Q_A^\dagger]$

- $[Q_A, Q_B^\dagger] = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underline{\underline{0}} = [Q_A^\dagger, Q_B]$