5. Homework in "Supersymmetric Quantum Mechanics" (SoSe 2023)

Problem 1: Some relations of Pauli matrices

The Pauli matrices σ_i, $i = 1,2,3$, obey the formula $\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k$, where ε_{ijk} is the Levi-Civita symbol also called epsilon tensor and summation over index k is implied.

a) Proof for arbitrary vector-valued operators $\vec{A}$ and $\vec{B}$ the following formula

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right) = \vec{A}\cdot\vec{B} + \mathrm{i}\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right)$$

b) Proof $(\vec{\sigma}\cdot\vec{e}_r)^2 = 1$, where $\vec{e}_r = \vec{r}/|\vec{r}|$, and $(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^2$.

c) Show that ($\hbar = 1$): $(\vec{\sigma}\cdot\vec{p}) = -\mathrm{i}\,(\vec{\sigma}\cdot\vec{e}_r)\left(\partial_r - \vec{\sigma}\cdot\vec{L}/r\right)$ Hint: $\vec{\sigma}\cdot\vec{p} = (\vec{\sigma}\cdot\vec{e}_r)^2\,\vec{\sigma}\cdot\vec{p}$

d) Show that ($\hbar = 1$): $[\vec{\sigma}\cdot\vec{e}_r, \vec{\sigma}\cdot\vec{p}] = 2\mathrm{i}\vec{\sigma}\cdot\vec{L}/r$

Problem 2: Representations of the Dirac algebra

The Dirac matrices $\vec{\alpha}$ and β are required to obey the Dirac algebra

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}\,, \qquad \{\alpha_i, \beta\} = 0\,, \qquad \beta^2 = 1\,.$$

Show that the following representations in terms of Pauli matrices obey the Dirac algebra

a) Pauli representation: $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

b) Weyl representation: $\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

c) SUSY representation: $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.

Problem 3: The Dirac Oscillator

The Dirac oscillator is characterised by the Hamiltonian

$$H_D = \begin{pmatrix} M_+ & A \\ A^\dagger & -M_- \end{pmatrix}\,, \quad A := c\vec{\sigma}\cdot(\vec{p} - \mathrm{i}m\omega\vec{r})\,, \quad M_\pm := mc^2\,, \quad \mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^4\,.$$

Show that the corresponding SUSY partner Hamiltonians are given by

$$H_\pm = \left[\frac{\vec{p}^2}{2m} + \frac{m}{2}\omega^2\vec{r}^2 \pm \hbar\omega\left(K + \frac{1}{2}\right)\right]\,, \quad \text{where} \quad K = \vec{\sigma}\cdot\vec{L}/\hbar + 1$$