4. Homework in "Supersymmetric Quantum Mechanics" (SoSe 2023)

Problem 1: Two Examples for the supersymmetric Fokker-Planck equation

For a given SUSY potential $\Phi : \mathbb{R} \to \mathbb{R}$ the decay rates and decay modes of the probability density $m_t^{\pm}(x, x_0)$ of the Fokker-Planck equation with drift potential $U_{\pm}(x) = \mp \int \mathrm{d}x \Phi(x)$ are given by the eigenvalues and eigenstates of the corresponding Fokker-Planck Hamilonian

$$H_{\pm}^{\mathrm{FP}} = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\Phi^2(x) \pm \frac{1}{2}\Phi'(x)\,.$$

Calculate for the two examples

$$\Phi_1(x) = a \operatorname{sgn} x\,, \qquad \text{and} \qquad \Phi_2(x) = a \tanh x\,, \qquad a > 0$$

a) The stationary distribution $P_{\mathrm{st}}(x)$ (normalization excluded) for U_- if it exists.

b) The drift potentials $U_{\pm}$ and the Fokker-Planck potentials $V_{\pm}^{\mathrm{FP}}$.

c) The lowest non-vanishing decay rate $\lambda_n > 0$.

Problem 2: Conditionally exactly solvable Fokker-Planck potentials

Assume that the eigenvalues and eigenstates of the Fokker-Planck Hamiltonian H_+^{FP} associated with a SUSY potential Φ are explicitly known. Consider a "perturbed" SUSY potential $W(x) = \Phi(x) + f(x)$ and show that for a function f obeying the following generalized Riccati equation

$$f^2(x) + 2\Phi(x)f(x) + f'(x) = b\,, \quad b = const.$$

the pair of the Fokker-Planck potentials associated with the SUSY potential W reads

$$\begin{aligned} V_+^{\mathrm{FP}}(x) &= \tfrac{1}{2}\Phi^2(x) + \tfrac{1}{2}\Phi'(x) + \tfrac{b}{2} \\ V_-^{\mathrm{FP}}(x) &= \tfrac{1}{2}\Phi^2(x) - \tfrac{1}{2}\Phi'(x) - f'(x) + \tfrac{b}{2}\,. \end{aligned}$$

Using the ansatz $f(x) = v'(x)/v(x)$ show that the generalized Riccati equation can be linearized and results in the second-order differential equation

$$v''(x) + 2\Phi(x)v'(x) - bv(x) = 0\,.$$

With the help of the substitution $u(x) = v(x) \exp\{\int \mathrm{d}x\, \Phi(x)\}$ show that this results in the Schrödinger-like equation

$$\left[-\tfrac{1}{2}\partial_x^2 + \tfrac{1}{2}\Phi^2(x) + \tfrac{1}{2}\Phi'(x)\right] u(x) = \varepsilon u(x)\,.$$

How are b and ε related? What conditions have to be obeyed by ε and u in order to have a well-defined and unbroken SUSY? Calculate the drift potentials $U_{\pm}$ in terms of u.