3. Homework in "Supersymmetric Quantum Mechanics" (SoSe 2023)

Problem 1: The harmonic oscillator SUSY partner for $\varepsilon = -5/2$ and $\beta = 0$

Consider the special member $\varepsilon = -5/2$ and $\beta = 0$ of the family of SUSY partners of the harmonic oscillator discussed in section 4.2 of the lectures. This case corresponds to the solution of the Schrödinger-like equation given by $u(x) = -(4x^2 + 2)\exp\{x^2/2\}$. Show that the corresponding partner potential explicitly reads

$$V_-(x) = \frac{x^2}{2} + \frac{8x^2}{2x^2+1} + \left(\frac{4x}{2x^2+1}\right)^2 - 5.$$

Problem 2: A family of SUSY partners for the free particle

Following section 4.2 consider the case of a free particle, that is, $V_+(x) = 0$ and show that for this case the general solution of the Schrödinger-like equation reads $u(x) = \mathrm{e}^{\kappa x} + \beta \mathrm{e}^{-\kappa x}$, where $\varepsilon = -\hbar^2\kappa^2/2m$, $\kappa \in \mathbb{R}$. What are the allowed values for the parameter β and ε? Show that the family of partner potentials of the free particle explicitly reads

$$V_-(x) = \frac{\hbar^2\kappa^2}{m}\left[\left(\frac{\mathrm{e}^{\kappa x} - \beta\mathrm{e}^{-\kappa x}}{\mathrm{e}^{\kappa x} + \beta\mathrm{e}^{-\kappa x}}\right)^2 - 1\right].$$

Problem 3: Conservation of the energy functional

Let $\phi(x,t)$ be a solution of the field equation $\partial_t^2\phi - \partial_x^2\phi = -\partial U/\partial\phi$ with boundary conditions $\lim_{x\to\pm\infty}\partial_t\phi = 0 = \lim_{x\to\pm\infty}\partial_x\phi$. Show that the energy functional

$$E[\phi] := \int_{-\infty}^{+\infty} \mathrm{d}x \left[\frac{1}{2}\left(\partial_t\phi\right)^2 - \left(\partial_x\phi\right)^2 + U(\phi)\right]$$

is conserved, that is, $\mathrm{d}E/\mathrm{d}t = 0$.

Problem 4: The double quadratic field model via SUSY

Consider the SUSY potential $W(x) = \operatorname{sgn} x$. Following the discussion of section 5.2 show that this SUSY potential induces a field potential of the form

$$U(\phi) = \frac{1}{2}\left(1 - |\phi|\right)^2.$$

Hint: $\partial_x \operatorname{sgn} x = 2\delta(x)$.