2. Homework in "Supersymmetric Quantum Mechanics" (SoSe 2023)

Problem 1: SUSY Transformation for Continuum States

Following the discussion of Exercise 5 consider now the case $\Phi_-^2 < E < \Phi_+^2$. That is for $x \to \pm\infty$ the eigenfunction associated with energy eigenvalue E are described by

$$\begin{aligned} \phi_E^\pm(x) &\sim \exp\{\mathrm{i}k_L(E)x\} + R^\pm(E)\exp\{-\mathrm{i}k_L(E)x\} \quad \text{for} \quad x \to -\infty \\ \phi_E^\pm(x) &\sim T^\pm(E)\exp\{-\kappa_R(E)x\} \quad \text{for} \quad x \to \infty \,. \end{aligned}$$

a) Show that the inverse penetration length is given by

$$\kappa_R(E) = \frac{1}{\hbar}\sqrt{2m(\Phi_+^2 - E)}$$

and the corresponding amplitudes for the two partner potentials are related by

$$T^+(E) = \frac{\Phi_+ - \sqrt{\Phi_+^2 - E}}{\Phi_- + \mathrm{i}\sqrt{E - \Phi_-^2}}\, T^-(E)\,.$$

b) Proof that $|T^+(E)|^2 < |T^-(E)|^2$ for $\Phi_-^2 < E < \Phi_+^2$.

Problem 2: Continuum states of the free particle and its SUSY partner

Consider the Witten model with SUSY potential $\Phi(x) = \frac{\hbar}{\sqrt{2m}}\tanh x$. See Exercise 6 a). Show that the corresponding continuum states are given by

$$\phi_E^+(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}kx}\,, \qquad \phi_E^-(x) = N(\tanh x - \mathrm{i}k)\mathrm{e}^{\mathrm{i}kx}\,,$$

where $E = \hbar^2(k^2+1)/2m$, $k \in \mathbb{R}$ and N is a proper normalisation constant.

Problem 3: Spectral properties of infinite square well and its SUSY partner

Consider the Witten model with SUSY potential $\Phi(x) = \frac{\hbar}{\sqrt{2m}}\tan x$ on $\mathcal{H}_\pm = L^2([-\pi/2, \pi/2])$ with Dirichlet boundary conditions $\phi_E^\pm(\pm\pi/2) = 0$. See Exercise 6 c). Show that

a) The eigenvalues and eigenstates of H_- are given by

$$E_n = \frac{\hbar^2}{2m}(n^2 - 1)\,, \qquad \phi_{E_n}^-(x) = \sqrt{\frac{2}{\pi}}\sin(nx + n\pi/2)\,, \qquad n = 1, 2, 3 \ldots.$$

b) The SUSY ground state $\phi_{E_1} = \phi_0$ is annihilated by the operator A.

c) The eigenstates of H_+ are given by ($n = 2, 3, 4, \ldots$)

$$\phi_{E_n}^+(x) = \sqrt{\frac{2}{\pi(n^2-1)}}\left(n\cos(nx + n\pi/2) + \tan x\,\sin(nx + n\pi/2)\right)\,.$$