1. Homework in "Supersymmetric Quantum Mechanics" (SoSe 2023)

Problem 1: The Free Particle on the Real Line $(N = 1)$

The free motion of a mass m on the real line is characterized by the Hamiltonian $H = p^2/2m$ with momentum operator $p = (\hbar/\mathrm{i})\partial_x$ acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$.

a) Show that $Q_1 := p/\sqrt{4m}$ is a supercharge and $\{H, Q_1; \mathcal{H}\}$ forms a supersymmetric quantum system with unbroken SUSY.

b) Let $|k\rangle$ denote the eigenstates of p, that is, $p|k\rangle = \hbar k|k\rangle$ with $k \in \mathbb{R}$ being the so-called wave number. Consider the generalised helicity operator $\Lambda := \operatorname{sgn} Q_1 = Q_1/|Q_1|$ and show that Λ induces a grading on $\mathcal{H}$. Why does Λ not represent a Witten operator?

Problem 2: SUSY Transformations for $N = 2$

Consider a SUSY quantum system $\{H, Q_1, Q_2; \mathcal{H}\}$ and let $|q\rangle$ be an eigenstate of the self-adjoint supercharge Q_1, that is, $Q_1|q\rangle = q|q\rangle$. Obviously $|q\rangle$ is also an eigenstate of H with eigenvalue $E = 2q^2$. Show that for $q \neq 0$ the state $Q_2|q\rangle$ is a simultaneous eigenstate of H and Q_1 with eigenvalues $E = 2q^2$ and $-q$, respectively.

Problem 3: The Witten Model

Consider a Cartesian degree of freedom on the real line which carries an additional internal spin-$\frac{1}{2}$-like degree of freedom. Hence, the Hilbert space is given by $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^2$. Define two real supercharges as follows:

$$Q_1 := \frac{1}{\sqrt{2}} \left(\frac{p}{\sqrt{2m}} \otimes \sigma_1 + \Phi(x) \otimes \sigma_2 \right), \quad Q_2 := \frac{1}{\sqrt{2}} \left(\frac{p}{\sqrt{2m}} \otimes \sigma_2 - \Phi(x) \otimes \sigma_1 \right),$$

with Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and Φ being a real-valued function, $\Phi : \mathbb{R} \to \mathbb{R}$, which is assumed to be continuously differentiable. Show that this system forms an $N = 2$ SUSY system with partner Hamiltonians given by

$$H_\pm := \frac{p^2}{2m} + \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \Phi'(x)\,.$$

Problem 4: On the Matrix Representation of Supercharges

Show for the operators as defined in Exercise 4 that all A-operators commute with all B-operators. That is, $[H_A, H_B] = 0$, $[H_A, Q_B] = 0 = [H_A, Q_b^\dagger]$, $[H_B, Q_A] = 0 = [H_B, Q_A^\dagger]$, $[Q_A, Q_B] = 0 = [Q_A^\dagger, Q_B^\dagger]$ and $[Q_A, Q_B^\dagger] = 0 = [Q_A^\dagger, Q_B]$.