Exercise 14: Spectrum Generating Algebra $su(1,1)$

The Group $SU(1,1)$

The set of 2×2 matrices

$$g = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \qquad \text{with} \qquad a, b \in \mathbb{C}\,, \qquad |a|^2 - |b|^2 = 1$$

forms the group $SU(1,1)$. The matrices are quasi-unitary as

$$g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Vilenkin uses notation $QU(2)$.

Parametrisation:

$$g(\alpha,\beta,\gamma) = \begin{pmatrix} \cosh\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} & \sinh\frac{\beta}{2}\,\mathrm{e}^{\mathrm{i}\frac{\gamma}{2}} \\ \sinh\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}} & \cosh\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} \end{pmatrix}$$

With $0 \leq \alpha, \gamma < 2\pi$ and $0 \leq \beta, \infty$

Generators:

$$X_\alpha = \left.\frac{\partial g}{\partial \alpha}\right|_e = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\mathrm{i}}{2}\sigma_3$$

$$X_\beta = \left.\frac{\partial g}{\partial \beta}\right|_e = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1$$

$$X_\gamma := [X_\alpha, X_\beta] = \frac{\mathrm{i}}{4}[\sigma_3,\sigma_1] = \frac{\mathrm{i}}{4}\,2\mathrm{i}\sigma_2 = -\frac{1}{2}\sigma_2$$

Algebra:

$$[X_\alpha, X_\beta] = X_\gamma\,, \qquad [X_\beta, X_\gamma] = -X_\alpha\,, \qquad [X_\gamma, X_\alpha] = X_\beta\,,$$

or with $J_1 := -\mathrm{i}X_\gamma$, $J_2 := -\mathrm{i}X_\beta$,$J_3 := -\mathrm{i}X_\alpha$

$$[J_1, J_2] = -\mathrm{i}J_3\,, \qquad [J_2, J_3] = \mathrm{i}J_1\,, \qquad [J_3, J_1] = \mathrm{i}J_2$$

Cartan metric:

$$g_{11} = c^3_{12}c^2_{13} = (-\mathrm{i})(-\mathrm{i}) = -1$$
$$g_{22} = c^1_{23}c^3_{21} = (\mathrm{i})(\mathrm{i}) = -1$$
$$g_{33} = c^2_{31}c^1_{32} = (\mathrm{i})(-\mathrm{i}) = 1$$

Casimir: $\vec{J}^2 = -J_1^2 - J_2^2 + J_3^2$ is NOT bounded!!!

$SU(1,1)$ is a non-compact group but locally compact

All UIR are infinite-dimensional

The UIR of $SU(1,1)$

Let us enumerate the UIR similar to $SU(2)$ by label j and chose basis digitalising the compact operator J_3

$$\vec{J}^2|jm\rangle = j(j+1)|jm\rangle\,, \qquad J_3|jm\rangle = m|jm\rangle$$

Note $j \leftrightarrow -j-1$ are equivalent reps

- *Continuous Principle Series:* $D_c^{(-\frac{1}{2}+\mathrm{i}\rho,\varepsilon_0)}$

$$j = -\tfrac{1}{2} + \mathrm{i}\rho\,, \qquad \rho > 0\,, \qquad \varepsilon_0 \in [-\tfrac{1}{2}, \tfrac{1}{2}[$$

$$\operatorname{spec}\vec{J}^2 = -\tfrac{1}{4} - \rho^2 < -\tfrac{1}{4}\,, \qquad \operatorname{spec} J_3 = \varepsilon_0 + m\,, \qquad m \in \mathbb{Z}$$

- *Continuous Supplementary Series:* $D_s^{(j,\varepsilon_0)}$

$$j \in [-\tfrac{1}{2}, 0]\,, \qquad \varepsilon_0 \in [-\tfrac{1}{2}, \tfrac{1}{2}] \qquad \text{with} \qquad |j + \tfrac{1}{2}| \le \tfrac{1}{2} - |\varepsilon_0|$$

$$\operatorname{spec} \vec{J}^2 = [-\tfrac{1}{4}, 0]\,, \qquad \operatorname{spec} J_3 = \varepsilon_0 + m\,, \qquad m \in \mathbb{Z}$$

- *Discrete Series:* D_j^+

$$j > -1\,, \qquad m = j+1, j+2, \dots \qquad \qquad \text{bounded below}$$

- *Discrete Series:* D_j^-

$$j > -1\,, \qquad m = -j-1, -j-2, \dots \qquad \qquad \text{bounded above}$$

The $su(1,1)$ symmetry of the $1/r$ problem in $\mathbb{R}^d$

$\mathcal{H} = L^2(\mathbb{R}^d)\,, \qquad [Q_i, P_j] = \mathrm{i}\delta_{ij} \qquad i,j = 1,2,\dots,d \ge 2\,, \qquad (\hbar = 1)$
$R := |\vec{Q}| = \big(Q_1^2 + Q_2^2 + \cdots + Q_d^2\big)^{1/2}$

The $su(1,1)$ structure:
Let

$$J_1 := \frac{1}{2}(R\vec{P}^2 - R)\,, \qquad J_2 := \vec{Q}\cdot\vec{P} - \mathrm{i}\frac{d-1}{2}\,, \qquad J_3 := \frac{1}{2}(R\vec{P}^2 + R)$$

then a little calculation shows that these obey an $su(1,1)$ algebra

$$[J_1, J_2] = -\mathrm{i}J_3\,, \qquad [J_2, J_3] = \mathrm{i}J_1\,, \qquad [J_3, J_1] = \mathrm{i}J_2$$

with Casimir operator

$$\vec{J}^2 = -J_1^2 - J_2^2 + J_3^2 = \vec{Q}^2\vec{P}^2 + \mathrm{i}(d-2)\vec{Q}\cdot\vec{P} - (\vec{Q}\cdot\vec{P})^2 + \frac{1}{4}(d-1)(d-3)$$

Angular momentum in $\mathbb{R}^d$:
Let

$$L_{ik} := Q_i P_k - Q_k P_i = -L_{ki}$$

then

$$\vec{L}^{\,2} := \frac{1}{2}\sum_{i,k=1}^{d} L_{ik}^2 = \vec{Q}^2\vec{P}^2 + \mathrm{i}(d-2)\vec{Q}\cdot\vec{P} - (\vec{Q}\cdot\vec{P})^2$$

Note $\vec{L}^{\,2}$ has eigenvalues $\ell(\ell+d-2)$, $\ell = 0,1,2,3,\dots$
Observation:

$$\vec{J}^2 = \vec{L}^{\,2} + \frac{1}{4}(d-1)(d-3)$$

Angular momentum eigenspace is also reps space of $su(1,1)$

$$\begin{aligned} j(j+1) &= \ell(\ell+d-2) + \tfrac{1}{4}(d-1)(d-3) \\ j^2 + j + \tfrac{1}{4} &= \ell^2 + \ell(\ell+d-2) + \tfrac{1}{4}(d^2 - 4d + 4) \\ (j+\tfrac{1}{2})^2 &= (\ell + \tfrac{d-2}{2})^2 \end{aligned}$$

Hence

$$\boxed{j = \ell + \tfrac{d-3}{2}}$$

Furthermore $J_3 \ge 0 \Rightarrow D_j^+$ series with integer or half-integer j for odd or even d.

The eigenvalue problem

$$H = \frac{\vec{P}^{\,2}}{2m} - \frac{\alpha}{R}$$

$$(H-E)|\psi\rangle = 0 \qquad \Longleftrightarrow \qquad R(H-E)|\psi\rangle = 0$$

Consider: $\Theta := R(H-E)$ with $\Theta|\psi\rangle = 0$

$$\Theta = \frac{1}{2m} R\vec{P}^{\,2} - \alpha - ER = \frac{1}{2m}(J_1+J_3) - E(J_3-J_1) - \alpha$$

Tilting:

$$\left.\begin{array}{l} \tilde{\Theta} := \mathrm{e}^{-\mathrm{i}\theta J_2}\,\Theta\,\mathrm{e}^{\mathrm{i}\theta J_2} \\ |\tilde{\psi}\rangle := \mathrm{e}^{-\mathrm{i}\theta J_2}|\psi\rangle \end{array}\right\} \tilde{\Theta}|\tilde{\psi}\rangle = 0$$

physical state: $|\psi\rangle$
group state: $|\tilde{\psi}\rangle$
Using

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\theta J_2} J_1 \mathrm{e}^{\mathrm{i}\theta J_2} &= J_1\cosh\theta + J_3\sinh\theta \\ \mathrm{e}^{-\mathrm{i}\theta J_2} J_3 \mathrm{e}^{\mathrm{i}\theta J_2} &= J_3\cosh\theta + J_1\sinh\theta \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \tilde{\Theta} &= \tfrac{1}{2m}\left(J_1\cosh\theta + J_3\sinh\theta + J_1\sinh\theta + J_3\cosh\theta\right) \\ &\quad - E\left(J_3\cosh\theta + J_1\sinh\theta - J_3\sinh\theta - J_1\cosh\theta\right) - \alpha \\ &= J_1\left(\cosh\theta(\tfrac{1}{2m}+E) + \sinh\theta(\tfrac{1}{2m}-E)\right) \\ &\quad + J_3\left(\sinh\theta(\tfrac{1}{2m}+E) + \cosh\theta(\tfrac{1}{2m}-E)\right) - \alpha \end{aligned}$$

Considering bound states $E<0$ we choose θ such that $\tilde{\Theta}$ is independent of J_1:

$$\cosh\theta\left(\frac{1}{2m}+E\right) = -\sinh\theta\left(\frac{1}{2m}-E\right)$$

$\Rightarrow$

$$\sinh\theta\left(\frac{1}{2m}+E\right) = -\cosh\theta\frac{(\frac{1}{2m}+E)^2}{\frac{1}{2m}-E}\,,$$

$$\tanh\theta = \frac{E+\frac{1}{2m}}{E-\frac{1}{2m}} \qquad \text{and} \qquad \cosh^2\theta = \frac{1}{1-\tanh^2\theta} = \frac{(E-\frac{1}{2m})^2}{-\frac{2E}{m}}$$

$$\begin{aligned} \tilde{\Theta} &= J_3\left(-\cosh\theta\frac{(\frac{1}{2m}+E)^2}{\frac{1}{2m}-E} + \cosh\theta(\frac{1}{2m}-E)\right) - \alpha \\ &= J_3\underbrace{\frac{\cosh\theta}{\frac{1}{2m}-E}}_{\sqrt{\frac{m}{-2E}}}\underbrace{\left(-\left(\frac{1}{2m}+E\right)^2 + \left(\frac{1}{2m}-E\right)^2\right)}_{-\frac{2E}{m}} - \alpha \\ &= J_3\sqrt{\frac{-2E}{m}} - \alpha \end{aligned}$$

Hence

$$\tilde{\Theta}|\tilde{\psi}\rangle = \left(J_3\sqrt{\frac{-2E}{m}} - \alpha\right)|\tilde{\psi}\rangle \overset{!}{=} 0$$

$\Rightarrow$

$$J_3|\tilde{\psi}\rangle = \alpha\sqrt{\frac{m}{-2E}}\,|\tilde{\psi}\rangle \qquad \text{eigenstate of } J_3$$

Hence, we choose $|\tilde{\psi}\rangle = |jn\rangle \in D_j^+$ with $J_3|jn\rangle = n|jn\rangle$, $n = j+1, j+2, \ldots$
Remember

$$j = \ell + \frac{d-3}{2} \qquad \Rightarrow \qquad n = \ell + \frac{d-1}{2} + n_r \qquad n_r = 0,1,2,\ldots$$

to obtain the eigenvalues

$$n = \alpha\sqrt{\frac{m}{-2E}} \qquad \Rightarrow \qquad E_n = -\frac{m\alpha^2}{2n^2}$$

and eigenstates

$$|\psi_n\rangle = \mathrm{e}^{\mathrm{i}\theta_n J_2}|\tilde{\psi}_n\rangle = \mathrm{e}^{\mathrm{i}\theta_n J_2}|\ell + \tfrac{d-3}{2}\,, n\rangle \qquad \text{with} \qquad \tanh\theta_n = \frac{E_n + \frac{1}{2m}}{E_n - \frac{1}{2m}}$$

Considering scattering states with $E > 0$ in essence we choose a basis diagonalising J_1:

$$J_1|j\lambda\rangle = \lambda|j\lambda\rangle \qquad \lambda \in \mathbb{R}$$

In essence same calculation again but with $J_3 \to J_1$
Result:

$$J_1|\tilde{\psi}\rangle = \alpha\sqrt{\frac{m}{2E}}\,|\tilde{\psi}\rangle\,, \qquad |\tilde{\psi}\rangle = |j\lambda\rangle$$

$$E_\lambda = \frac{m\alpha^2}{2\lambda^2} \geq 0\,, \qquad |\psi_\lambda\rangle = \mathrm{e}^{\mathrm{i}\theta_\lambda J_2}|\ell + \tfrac{d-3}{2}\,, \lambda\rangle \qquad \text{with} \qquad \tanh\theta_\lambda = \frac{E_\lambda - \frac{1}{2m}}{E_\lambda + \frac{1}{2m}}$$

The $su(1,1)$ symmetry of the harmonic oscillator in $\mathbb{R}^d$

Let

$$J_1 := \frac{1}{4}\sum_{i=1}^{d}\left((a_i^\dagger)^2 + a_i^2\right)\,, \qquad J_2 := -\frac{\mathrm{i}}{4}\sum_{i=1}^{d}\left((a_i^\dagger)^2 - a_i^2\right)\,, \qquad J_3 := \frac{1}{2}\sum_{i=1}^{d}\left(a_i^\dagger a_i + \frac{1}{2}\right)\,,$$

With help of

$$[a_i, a_j^\dagger] = \delta_{ij}$$

show that the J_i's close $su(1,1)$ algebra

$$[J_1, J_2] = -\mathrm{i}J_3\,, \qquad [J_2, J_3] = \mathrm{i}J_1\,, \qquad [J_3, J_1] = \mathrm{i}J_2$$

Hamiltonian:

$$H = \hbar\omega\sum_{i=1}^{d}\left(a_i^\dagger a_i + \frac{1}{2}\right) = 2\hbar\omega\, J_3 > 0$$

Casimir operator:

$$\vec{L}^{\,2} = 4\vec{J}^2 - \frac{d}{4}(d-4)$$

$\Rightarrow$

$$\ell(\ell + d - 2) = 4j(j+1) - \frac{d}{4}(d-4) \qquad \Rightarrow \qquad j = \frac{\ell}{2} + \frac{d}{4} - 1$$

Hence we have the reps D_j^+

Eigenstates are those of J_3:

$$J_3|jm\rangle = m|jm\rangle \qquad \text{with} \qquad m = j + 1 + n_r = \frac{\ell}{2} + \frac{d}{4} + n_r\,, \qquad n_r \in \mathbb{N}_0$$

Eigenvalues:

$$E_{n_r} = 2\hbar\omega m = \hbar\omega\left(2n_r + \ell + \frac{d}{2}\right)$$

Lit.: A.O. Barut *Dynamical Groups and Generalized Symmetries in Quantum Theory* (Univ. Canterbury, Christchurch, NZ,1972)

*** End of Tutorial 5 ***