Exercise 11: Wigner's construction of left-invariant measure for Lie groups

Assume a left-invariant measure exists: $\mathrm{d}\mu(g) = \mathrm{d}\mu(g_0^{-1} g)$
Then $g = g(\alpha_1, \ldots, \alpha_n)$ and, with g_0 fixed, $g_0^{-1} g = g_1 = g(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon = \varepsilon(\alpha)$

Invariance implies

$$\rho(\alpha_1, \ldots, \alpha_n)\mathrm{d}^n\alpha = \rho(\varepsilon_1, \ldots, \varepsilon_n)\mathrm{d}^n\varepsilon$$

$\Rightarrow$

$$\rho(\varepsilon_1, \ldots, \varepsilon_n) = \rho(\alpha_1, \ldots, \alpha_n)\frac{\partial(\varepsilon_1, \ldots, \varepsilon_n)}{\partial(\alpha_1, \ldots, \alpha_n)}$$

Let $P_k(g) := \alpha_k$ function which provides the k-th parameter of group element g
Idea: Let ε_i be very small, that is, $g_1 = g(\varepsilon_1, \ldots, \varepsilon_n) \sim e$ is very close to neutral element
Then $g = g_0 g_1 = g(\alpha)g(\varepsilon)$

$$\boxed{\rho(\alpha_1, \ldots, \alpha_n) = \rho_0 \lim_{\varepsilon\to 0}\left[\frac{\partial(P_1(g(\alpha)g(\varepsilon), \ldots, P_n(g(\alpha)g(\varepsilon)))}{\partial(\varepsilon_1, \ldots, \varepsilon_n)}\right]^{-1}}$$

Problem is reduced to expand functions P_k for small ε
$\rho_0 = \rho(0, \ldots, 0)$ is density of matrix elements near neutral element $\Leftrightarrow$ normalisation

Example: $G = SO(2)$

$$g = g(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad g = g(\varepsilon) = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}$$

$$P(g(\alpha)g(\varepsilon)) = P(g(\alpha+\varepsilon)) = \alpha + \varepsilon \qquad \Rightarrow \qquad \frac{\partial P}{\partial\varepsilon} = 1$$

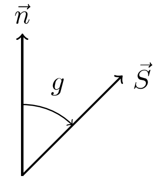
$$\Rightarrow \qquad \rho(\alpha) = \rho_0 := \frac{1}{2\pi} \qquad \Rightarrow \qquad \mathrm{d}\mu(g) = \frac{1}{2\pi}\mathrm{d}\alpha$$

In general and in particular for non-abelian groups the problem is to find an explicit form for functions $P_k(g(\alpha)g(\varepsilon))$.

Exercise 12: Stanley's n-vector model on the linear chain

"Spin":

$\vec{S} \in S^{n-1}$ "spin space"
$SO(n)$ transitive transformation group
Let $\vec{S} = g\vec{n}$, $\vec{n}$ = north pole, $g \in SO(n)$
$SO(n-1) = \{g \in SO(n) | g\vec{n} = \vec{n}\}$ massive



Invariant Haar measure dg: $\int_{SO(n)} \mathrm{d}g = 1$

Invariant normalised measure on S^{n-1}:

$$\mathrm{d}\mu(\vec{S}) = \frac{\Gamma(n/2)}{2\pi^{n/2}} \mathrm{d}^{n-1}\Omega(\vec{S})$$

is a priori normalised probability distribution of $\vec{S}$

"Spin interaction":

Shall be $SO(n)$ invariant and exchange invariant

$$V(\vec{S}, \vec{S}') = V(g\vec{S}, g\vec{S}') = V(\vec{S}', \vec{S})$$

For example: $V(\vec{S}, \vec{S}') = -J\vec{S} \cdot \vec{S}' - K(\vec{S} \cdot \vec{S}')^2$
$K = 0$: Stanley's n-vector model
$K > 0$: Stanley's n-vector model on harmonic chain

Consider:

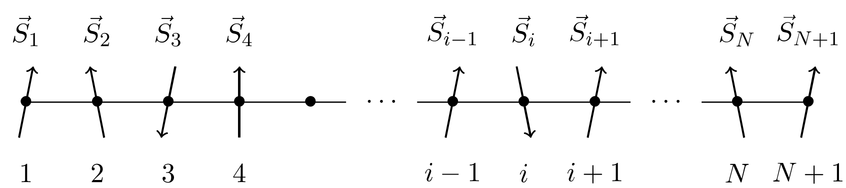
$$v(g) := V(g\vec{n}, \vec{n}) = V(\vec{n}, g\vec{n}) = v(g^{-1})$$

is a zonal spherical function as for all $h, h' \in SO(n-1)$

$$v(hgh') = V(hgh'\vec{n}, \vec{n}) = V(hg\vec{n}, \vec{n}) = V(g\vec{n}, h^{-1}\vec{n}) = V(g\vec{n}, \vec{n}) = v(g)$$

Linear chain:

Chain of $N+1$ sites, each occupied by a "spin" $\vec{S}_j := g_j\vec{n}$, $g_j \in SO(n)$



Nearest neighbour interaction:

$$H := \sum_{j=1}^{N} V(\vec{S}_j, \vec{S}_{j+1}) = \sum_{j=1}^{N} v(g_j^{-1} g_{j+1})$$

Partition function:

$$\begin{aligned} Z(\beta) &:= \int_{S^{n-1}} \mathrm{d}\mu(\vec{S}_1) \cdots \int_{S^{n-1}} \mathrm{d}\mu(\vec{S}_{N+1})\, \mathrm{e}^{-\beta H}, \qquad \beta := 1/k_B T \\ &= \int_{SO(n)} \mathrm{d}g_1 \cdots \int_{SO(n)} \mathrm{d}g_{N+1} \prod_{j=1}^{N} \exp\left\{-\beta v(g_j^{-1} g_{j+1})\right\} \end{aligned}$$

Harmonic analysis:

Class 1 reps of $SO(n)$: $\ell \in \Lambda = \{0, 1, 2, 3, \ldots\}$, $\qquad d_\ell = (2\ell + n - 2)\dfrac{\Gamma(\ell + n - 2)}{\Gamma(\ell+1)\Gamma(n-1)}$

$L^2(S^{n-1}) = \sum_{\ell=0}^{\infty} \mathcal{H}^\ell, \qquad \dim \mathcal{H}^\ell = d_\ell$

As $v(g)$ is zonal spherical function

$$\begin{aligned} \mathrm{e}^{-\beta v(g)} &= \sum_{\ell=0}^{\infty} d_\ell\, \lambda_\ell(\beta)\, D_{00}^{\ell}(g) \\ \lambda_\ell(\beta) &= \int_{SO(n)} \mathrm{d}g\, \mathrm{e}^{-\beta v(g)}\, D_{00}^{\ell\,*}(g) \end{aligned}$$

Hence

$$\begin{aligned} Z(\beta) = \quad & \int_{SO(n)} \mathrm{d}g_1 \cdots \int_{SO(n)} \mathrm{d}g_{N+1} \sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_N=0}^{\infty} d_{\ell_1} \cdots d_{\ell_N}\, \lambda_{\ell_1}(\beta) \cdots \lambda_{\ell_N}(\beta) \\ & \times D_{00}^{\ell_1}(g_1^{-1} g_2) \cdots D_{00}^{\ell_N}(g_N^{-1} g_{N+1}) \end{aligned}$$

Orthogonality:

$$\begin{aligned} & \int_{SO(n)} \mathrm{d}g_j\, D_{00}^{\ell_{j-1}}(g_{j-1}^{-1} g_j) D_{00}^{\ell_j}(g_j^{-1} g_{j+1}) \\ & \quad = \sum_{mn} D_{0m}^{\ell_{j-1}}(g_{j-1}^{-1}) \underbrace{\int_{SO(n)} \mathrm{d}g_j\, D_{m0}^{\ell_{j-1}}(g_j) D_{n0}^{\ell_j\,*}(g_j)}_{=\delta_{\ell_{j-1}\ell_j}\delta_{mn}/d_{\ell_j}} D_{00}^{\ell_j}(g_{j+1}) \\ & \quad = \frac{\delta_{\ell_{j-1}\ell_j}}{d_{\ell_j}} \sum_m D_{0m}^{\ell_{j-1}}(g_{j-1}^{-1}) D_{m0}^{\ell_j}(g_{j+1}) \\ & \quad = \frac{\delta_{\ell_{j-1}\ell_j}}{d_{\ell_j}} D_{00}^{\ell_j}(g_{j-1}^{-1} g_{j+1}) \end{aligned}$$

$\Rightarrow$

$$Z(\beta) = \int_{SO(n)} \mathrm{d}g_1 \int_{SO(n)} \mathrm{d}g_{N+1} \sum_{\ell=0}^{\infty} d_\ell\, [\lambda_\ell(\beta)]^N\, D_{00}^{\ell}(g_1^{-1} g_{N+1})$$

For open boundaries:

$$\begin{aligned} & \int_{SO(n)} \mathrm{d}g_{N+1}\, D_{00}^{\ell}(g_1^{-1} g_{N+1}) = \int_{SO(n)} \mathrm{d}g\, D_{00}^{\ell}(g) = \delta_{\ell 0} \\ & \int_{SO(n)} \mathrm{d}g_1 = 1 \end{aligned}$$

Result:

$$\boxed{Z(\beta) = \lambda_0^N(\beta)}$$

with

$$\lambda_0(\beta) = \int_{SO(n)} \mathrm{d}g\, \mathrm{e}^{-\beta v(g)} = \int_{S^{n-1}} \mathrm{d}\mu(\vec{S}) \exp\{-\beta V(\vec{S}, \vec{n})\}$$

For fixed boundaries: $\vec{S}_1 = g_1 \vec{n}$ and $\vec{S}_{N+1} = g_{N+1} \vec{n}$

$$\begin{aligned} Z_{\mathrm{fix}}(\beta) &= \sum_{\ell=0}^{\infty} d_\ell\, [\lambda_\ell(\beta)]^N\, D_{00}^{\ell}(g_1^{-1} g_{N+1}) \\ &= Z(\beta) \left(1 + \sum_{\ell \neq 0}^{\infty} d_\ell \left[\frac{\lambda_\ell(\beta)}{\lambda_0(\beta)} \right]^N D_{00}^{\ell}(g_1^{-1} g_{N+1}) \right) \end{aligned}$$

For periodic boundaries: $\vec{S}_1 = \vec{S}_{N+1} \Rightarrow g_1 = g_{N+1} \Rightarrow D^\ell_{00}(g_1^{-1} g_{N+1}) = D^\ell_{00}(e) = 1$

$$Z_{\text{fix}}(\beta) = Z(\beta) \left(1 + \sum_{\ell \neq 0}^{\infty} d_\ell \left[\frac{\lambda_\ell(\beta)}{\lambda_0(\beta)} \right]^N \right)$$

Large N limit:

Unitarity: $|D^\ell_{00}(g)| \leq D^0_{00}(g) = 1$
For $\ell \neq 0$ we may conclude $|\lambda_\ell(\beta)| < \int \mathrm{d}g\, \mathrm{e}^{-\beta v(g)} = \lambda_0(\beta)$ for all $\beta \Rightarrow \left| \frac{\lambda_\ell(\beta)}{\lambda_0(\beta)} \right| < 1$

Free energy:

$$F(\beta) := -\frac{1}{\beta} \lim_{N \to \infty} \frac{1}{N+1} \ln Z(\beta) = -\frac{1}{\beta} \ln \lambda_0(\beta)$$

Result is valid for all classical spin models with "spin space" = G/H with H massive and G-invariant and exchange invariant spin-spin interaction between nearest neighbours.

Explicit results for n-vector model

Assume $V(\vec{S}, \vec{S}') = f(\vec{S} \cdot \vec{S}')$, which is most general form of invariant interaction
$\Rightarrow V(g\vec{n}, \vec{n}) = v(g) = f(\cos\theta)$

$$D^\ell_{00}(g) = \frac{\Gamma(\ell+1)\Gamma(n-2)}{\Gamma(\ell+n-2)} C_\ell^{\frac{n-2}{2}}(\cos\theta), \qquad D^1_{00}(g) = \cos\theta = g\vec{n} \cdot \vec{n}$$

$$\begin{aligned} \lambda_\ell(\beta) &= \int_{SO(n)} \mathrm{d}g \mathrm{e}^{-\beta v(g)} D^\ell_{00}(g) \\ &= \frac{\Gamma(n/2)\Gamma(\ell+1)\Gamma(n-2)}{\sqrt{\pi}\Gamma(\frac{n-2}{2})\Gamma(\ell+n-2)} \int_0^\pi \mathrm{d}\theta\, \sin^{n-2}\theta\, \mathrm{e}^{-\beta f(\cos)}\, C_\ell^{\frac{n-2}{2}}(\cos\theta) \end{aligned}$$

In particular:

$$\lambda_0(\beta) = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^1 \mathrm{d}t\, \mathrm{e}^{-\beta f(t)} (1-t^2)^{\frac{n-3}{2}}$$

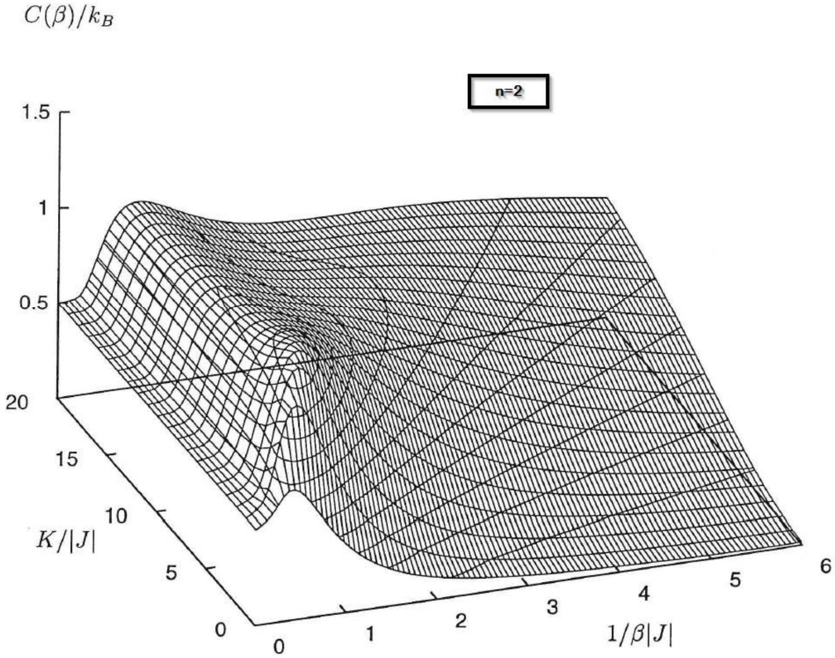
$$\lambda_1(\beta) = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^1 \mathrm{d}t\, \mathrm{e}^{-\beta f(t)} t (1-t^2)^{\frac{n-3}{2}}$$

More results:

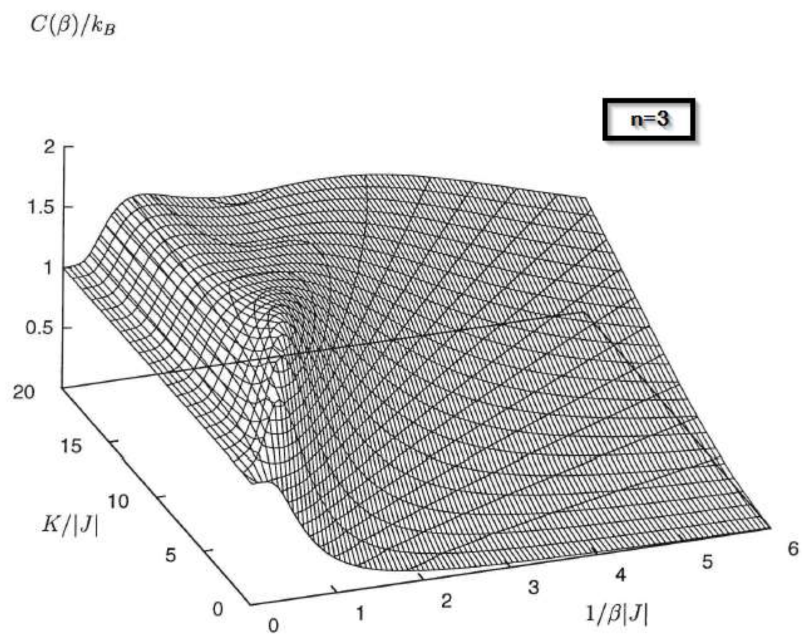
Free energy: $F(\beta) = -\frac{1}{\beta} \ln \lambda_0(\beta)$
Internal energy: $\frac{\partial}{\partial\beta}(\beta F(\beta)) = -\lambda_0(\beta)/\lambda_0'(\beta)$
Entropy: $S(\beta) = k_B \beta (E(\beta) - F(\beta)) = k_B \left(\ln \lambda_0(\beta) - \beta \frac{\lambda_0'(\beta)}{\lambda_0(\beta)} \right)$
Heat capacity: $c(\beta) = -k_B \beta^2 \frac{\partial E(\beta)}{\partial \beta} = k_B \beta^2 \left[\frac{\lambda_0''(\beta)}{\lambda_0(\beta)} - \left(\frac{\lambda_0'(\beta)}{\lambda_0(\beta)} \right)^2 \right]$
Spin correlation: $\langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle = \langle D^1_{00}(g_i^{-1} g_{i+r}) \rangle = \left(\frac{\lambda_1(\beta)}{\lambda_0(\beta)} \right)^r$
Zero-field susceptibility: $\chi_0(\beta) = \beta \left(1 + 2 \sum_{r=1}^{\infty} \langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle \right) = \beta \frac{1 + \frac{\lambda_1(\beta)}{\lambda_0(\beta)}}{1 - \frac{\lambda_1(\beta)}{\lambda_0(\beta)}}$

Next pages explicit results for harmonic chain with $V(\vec{S}, \vec{S}') = -J\vec{S} \cdot \vec{S}' - K(\vec{S} \cdot \vec{S}')^2$

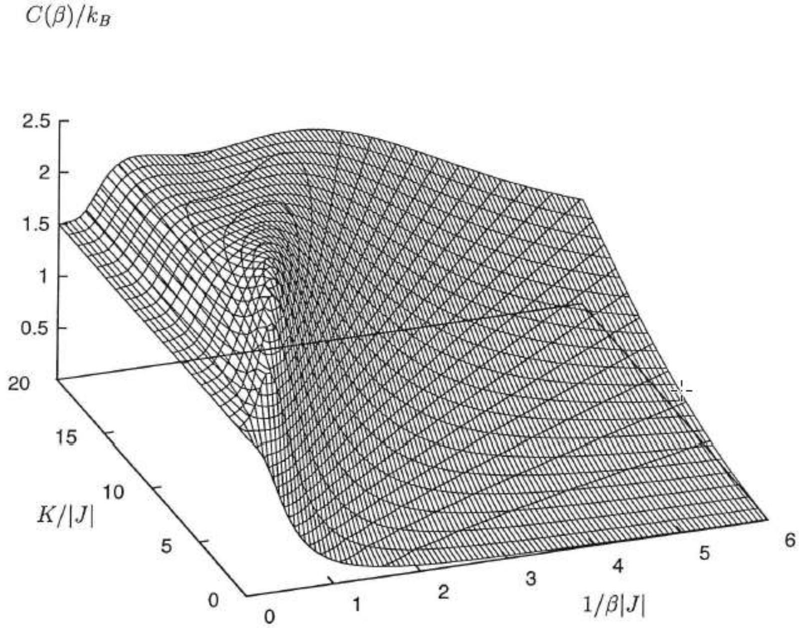
and "Kicked Rotator" (classical and quantum chaos)



$C(\beta)/k_B$
n=2
1.5
1
0.5
20
15
10
$K/|J|$
5
0
0
1
2
3
4
5
6
$1/\beta|J|$



$C(\beta)/k_B$
n=3
2
1.5
1
0.5
20
15
10
$K/|J|$
5
0
0
1
2
3
4
5
6
$1/\beta|J|$


$C(\beta)/k_B$
2.5
2
1.5
1
0.5
20
15
10
$K/|J|$
5
0
0
1
2
3
4
5
6
$1/\beta|J|$

Exercise 13: Wigner polynomials

Consider d_j dimensional UIR of $SU(2)$ via Euler angles acting in $\mathbb{C}^{2j+1}$

$$D^{(j)}(\varphi,\theta,\psi) := \mathrm{e}^{-\mathrm{i}\varphi J_3}\mathrm{e}^{-\mathrm{i}\theta J_2}\mathrm{e}^{-\mathrm{i}\psi J_3}$$

In standard basis: $J_3|jm\rangle = m|jm\rangle$ with $(J_1^2+J_2^2+J_3^2)|jm\rangle = j(j+1)|jm\rangle$
Matrix elements of D^j:

$$\langle jm|D^{(j)}(\varphi,\theta,\psi)|jn\rangle = \mathrm{e}^{-\mathrm{i}m\varphi}d^{(j)}_{mn}(\theta)\mathrm{e}^{-\mathrm{i}n\psi}$$

with the *Wigner polynomials*

$$d^{(j)}_{m,n}(\theta) := \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle$$

Problem: Find explicit form of Wigner polynomials

Solution:

$$\begin{aligned} d^{(j)}_{m,n}(\theta) &= \sqrt{\begin{pmatrix} j+m \\ j-n \end{pmatrix}\begin{pmatrix} j-n \\ j-m \end{pmatrix}}\cos^{m+n}\tfrac{\theta}{2}\sin^{m-n}\tfrac{\theta}{2}\,{}_2F_1(-j+m,j+m+1;m-n;\sin^2\tfrac{\theta}{2}) \\ &= \sqrt{\frac{(j+m)!(j-m)!}{(j+n)!(j-n)!}}\cos^{m+n}\frac{\theta}{2}\sin^{m-n}\frac{\theta}{2}P^{(m-n,m+n)}_{j-m}(\cos\theta) \end{aligned}$$

${}_2F_1(a,b;c;z)$ is the hypergeometric function
$P^{(m,n)}_l(z)$ is Jacobi polynomial

Properties:

- $d^{(j)}_{m,n}(\theta) = (-1)^{m-n}d^{(j)}_{-m,-n}(\theta) = (-1)^{m-n}d^{(j)}_{n,m}(\theta)$
- $d^{(j)}_{m,0}(\theta) = (-1)^m d^{(j)}_{0,m}(\theta) = \sqrt{\frac{(j-m)!}{(j+m)!}}P^m_j(\cos\theta)$ ass. Legendre Polynomial
- $d^{(j)}_{0,0}(\theta) = P_j(\cos\theta)$ Legendre Polynomial
- $Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}d^{(l)}_{m,0}(\theta)\mathrm{e}^{\mathrm{i}m\varphi}$ here $l = 0,1,2,3,\ldots$

Explicitly: In order $(\frac{1}{2},-\frac{1}{2})$ and $(1,0-1)$ from left to right and top to bottom

$$d^{(1/2)}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$d^{(1)}(\theta) = \begin{pmatrix} \cos^2\frac{\theta}{2} & -\frac{1}{\sqrt{2}}\sin\theta & \sin^2\frac{\theta}{2} \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ \sin^2\frac{\theta}{2} & \frac{1}{\sqrt{2}}\sin\theta & \cos^2\frac{\theta}{2} \end{pmatrix}$$

More see: A. Lindner *Drehimpulse in der QM* (Teubner, Stuttgart, 1984)

Proof:
Step 1:

$$\frac{\partial^2}{\partial\theta^2} d^{(j)}_{m,n}(\theta) = -\langle jm|J_2^2 \mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle$$

Step 2:

$$\begin{aligned} j(j+1)d^{(j)}_{m,n}(\theta) &= \langle jm|\vec{J}^2\, \mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle \\ &= \langle jm|(J_1^2+J_2^2+J_3^2)\, \mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle \\ &= \langle jm|J_1^2\, \mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle - \tfrac{\partial^2}{\partial\theta^2} d^{(j)}_{m,n}(\theta) + m^2 d^{(j)}_{m,n}(\theta) \end{aligned}$$

Step 3:
Noting that (see Addendum Exercise 10)

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\theta J_2} J_1^2\, \mathrm{e}^{-\mathrm{i}\theta J_2} = \; & (J_1\cos\theta + J_3\sin\theta)^2 \\ &= J_1^2\cos^2\theta + J_3^2\sin^2\theta + (J_1J_3+J_3J_1)\sin\theta\cos\theta \\ &= (\vec{J}^2 - J_2^2 - J_3^2)\cos^2\theta + J_3^2\sin^2\theta + (2J_1J_3 + \mathrm{i}J_2)\sin\theta\cos\theta \end{aligned}$$

$$\begin{aligned} \langle jm|J_1^2\, \mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle &= \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}\mathrm{e}^{\mathrm{i}\theta J_2} J_1^2\, \mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle \\ &= \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}(\vec{J}^2 - J_2^2 - J_3^2)\cos^2\theta|jn\rangle \\ &\quad + \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2} J_3^2|jn\rangle\sin^2\theta \\ &\quad + \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}(2J_1J_3+\mathrm{i}J_2)|jn\rangle\sin\theta\cos\theta \\ &= \cos^2\theta\, j(j+1)d^{(j)}_{m,n}(\theta) + \cos^2\theta\tfrac{\partial^2}{\partial\theta^2}d^{(j)}_{m,n}(\theta) - n^2\cos^2\theta d^{(j)}_{m,n}(\theta) \\ &\quad + n^2\sin^2\theta d^{(j)}_{m,n}(\theta) \\ &\quad + 2n\sin\theta\cos\theta\langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}J_1|jn\rangle - \sin\theta\cos\theta\tfrac{\partial}{\partial\theta}d^{(j)}_{m,n}(\theta) \end{aligned}$$

Step 4:

$$\begin{aligned} m d^{(j)}_{m,n}(\theta) &= \langle jm|J_3\mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle \\ &= \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}\mathrm{e}^{\mathrm{i}\theta J_2}J_3\mathrm{e}^{-\mathrm{i}\theta J_2}|jn\rangle \\ &= \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}(J_3\cos\theta - J_1\sin\theta)|jn\rangle \\ &= n\cos\theta d^{(j)}_{m,n} - \sin\theta\langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}J_1|jn\rangle \end{aligned}$$

Putting all together leads to a differential equation for the Wigner polynomial

$$\begin{aligned} j(j+1)d^{(j)}_{m,n}(\theta) &= -\tfrac{\partial^2}{\partial\theta^2}d^{(j)}_{m,n}(\theta) + m^2 d^{(j)}_{m,n}(\theta) \\ &\quad + \cos^2\theta\, j(j+1)d^{(j)}_{m,n}(\theta) + \cos^2\theta\tfrac{\partial^2}{\partial\theta^2}d^{(j)}_{m,n}(\theta) - n^2\cos^2\theta d^{(j)}_{m,n}(\theta) \\ &\quad + n^2\sin^2\theta d^{(j)}_{m,n}(\theta) - \sin\theta\cos\theta\tfrac{\partial}{\partial\theta}d^{(j)}_{m,n}(\theta) \\ &\quad + 2n\sin\theta\cos\theta\left(n\cos\theta d^{(j)}_{m,n} - m d^{(j)}_{m,n}(\theta)\right) \end{aligned}$$

$$\left[j(j+1)\sin^2\theta + \sin^2\theta\tfrac{\partial^2}{\partial\theta^2} + \sin\theta\cos\theta\tfrac{\partial}{\partial\theta} - m^2 + n^2\cos^2\theta - n^2\sin^2\theta - 2n^2\cos^2\theta + 2nm\cos\theta\right] d^{(j)}_{m,n}(\theta) = 0$$

$$\left[\tfrac{\partial^2}{\partial\theta^2} + \cot\theta\tfrac{\partial}{\partial\theta} - \tfrac{m^2+n^2-2nm\cos\theta}{\sin^2\theta} + j(j+1)\right] d^{(j)}_{m,n}(\theta) = 0$$

$$\left[\tfrac{\partial^2}{\partial\theta^2} + \cot\theta\tfrac{\partial}{\partial\theta} - \tfrac{(\frac{m-n}{2})^2}{\sin^2\frac{\theta}{2}} - \tfrac{(\frac{m+n}{2})^2}{\cos^2\frac{\theta}{2}} + j(j+1)\right] d^{(j)}_{m,n}(\theta) = 0$$

Hypergeometric differential eq. having Jacobi-polynomials as solution
See also QM problem for Pöschl-Teller potential

*** End of Tutorial 4 ***