Exercise 7: Classes and Character table of D_4

The group table: $d^4 = e = s^2$, $d^k s = d^{4-k} s$

D_4	e	d	d^2	d^3	s	ds	d^2s	d^3s
e	e	d	d^2	d^3	s	ds	d^2s	d^3s
d	d	d^2	d^3	e	ds	d^2s	d^3s	s
d^2	d^2	d^3	e	d	d^2s	d^3s	s	ds
d^3	d^3	e	d	d^2	d^3s	s	ds	d^2s
s	s	d^3s	d^2s	ds	e	d^3	d^2	d
ds	ds	s	d^3s	d^2s	d	e	d^3	d^2
d^2s	d^2s	ds	s	d^3s	d^2	d	e	d^3
d^3s	d^3s	d^2s	ds	s	d^3	d^2	d	e

Center of D_4: $Z(D_4) = \{e, d^2\} \Rightarrow$ each is a class by itself $\{e\}$, $\{d^2\}$
Consider element d: $\Rightarrow \{d, d^3\}$
Consider element s: $\Rightarrow \{s, d^2 s\}$
Remaining elements: $\Rightarrow \{ds, d^3 s\}$
Conclusion: D_4 has 5 classes given by

$$\{e\}\,, \{d^2\}\,\{d, d^3\}\,, \{s, d^2 s\}\,, \{ds, d^3 s\}$$

For the general case D_n see Homework 3 Problem 6a)
For the character table D_4 see Homework 3 Problem 6a)

D_4	$\{e\}$	$\{d^2\}$	$\{d, d^3\}$	$\{s, d^2 s\}$	$\{ds, d^3 s\}$
$D^{(1,1)}$	1	1	1	1	1
$D^{(1,2)}$	1	1	1	-1	-1
$D^{(1,3)}$	1	1	-1	1	-1
$D^{(1,4)}$	1	1	-1	-1	1
$D^{(2,1)}$	2	-2	0	0	0

Center of a Group:

$$Z(G) := \{z \in G | zg = gz\,, \forall g \in G\}$$

Set of all group elements communting with all elements of G.

Examples:

- $Z(D_n) = \{e\}$ for n odd
- $Z(D_n) = \{e, d^{n/2}\}$ for n even
- $Z(SU(2)) = \{-E, E\}$ $\qquad E = 2 \times 2$ unit matrix

Exercise 8: The Group $SU(2)$

Unitary complex 2×2 matrices with unit determinant.
General ansatz:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}$$

Unitarity, $gg^\dagger = 1 = g^\dagger g$, implies:
$|a|^2 + |b|^2 = 1$, $|c|^2 + |d|^2 = 1$, $|a|^2 + |c|^2 = 1$, $|b|^2 + |d|^2 = 1$ and $a^* c + b d^* = 0$, $ab^* + cd^* = 0$

$\det g = 1$ implies: $ad - cb = 1$

Hence:
$|b|^2 = |c|^2$ and $|a|^2 = |d|^2 \qquad \Rightarrow \qquad$ Ansatz: $d = a^* e^{i\alpha}$ and $c = -b^* e^{i\beta}$
$\Rightarrow \qquad ab^* + cd^* = ab^* - b^* a e^{i(\beta - \alpha)} = 0 \qquad \Rightarrow \alpha = \beta$
$\Rightarrow \qquad ad - cb = |a|^2 e^{i\alpha} + |b|^2 e^{i\alpha} = 1 \qquad \Rightarrow \alpha = 0$
$\Rightarrow \qquad d = a^*$ and $c = -b^*$ with $|a|^2 + |b|^2 = 1$

Result:

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad a, b \in \mathbb{C} \qquad \text{with} \qquad |a|^2 + |b|^2 = 1$$

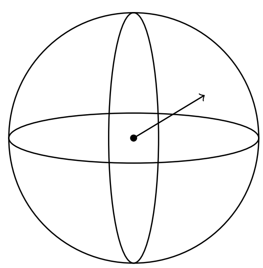
Ansatz:

$$a = \sqrt{1 - \vec{\alpha}^2} + i\alpha_3\,, \qquad b = \alpha_2 + i\alpha_1$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ with $|\vec{\alpha}|^2 \leq 1$.

Comments:

- Group space of $SU(2)$ is unit ball $B^3 := \left\{ \vec{\alpha} \in \mathbb{R}^3 \,\middle|\, |\vec{\alpha}|^2 \leq 1 \right\}$



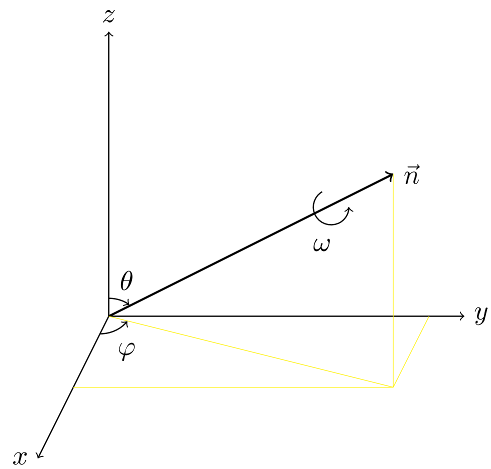
origin $\vec{0}$ represents the unit element

- Group space of $SU(2)$ is the unit sphere $S^3 := \left\{ \vec{x} \in \mathbb{R}^4 \,\middle|\, |\vec{x}|^2 = 1 \right\}$
  Set $\vec{x} = (\mathrm{Re}\, a, \mathrm{Im}\, a, \mathrm{Re}\, b, \mathrm{Im}\, b)$ with $|\vec{x}^2| = |a|^2 + |b|^2 = 1$
- $SU(2)$ is a simply connected space = every loop can be contracted to a point
- $SU(2)$ is the universal covering group of $SO(3)$ (see next exercise)

Exercise 9: The Group $SO(3)$

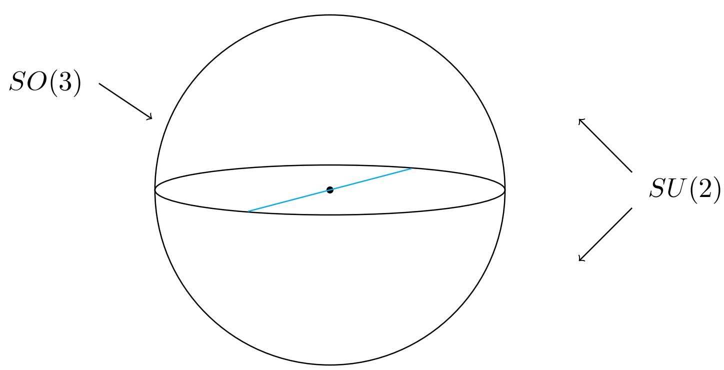
Group of rotations in $\mathbb{R}^3$ characterised by rotation axis $\vec{n} = \vec{n}(\theta, \varphi)$ and rotation angle $\omega \in [0, 2\pi[$:

$$R : \begin{array}{l} \mathbb{R}^3 \to \mathbb{R}^3 \\ \vec{x} \mapsto R\vec{x} \end{array} \qquad \text{with} \qquad |R\vec{x}|^2 = |\vec{x}|^2 \qquad \text{and} \qquad \det R = 1$$



Note: Rotation around $\vec{n}$ by angle ω is equivalent to rotation around $-\vec{n}$ and angle $2\pi - \omega$
$\Rightarrow$ For $SO(3)$ only upper half of unit sphere formed by $\vec{n}$ and only half of the equator

Group space of $SO(3)$: $\vec{\omega} = \omega\vec{n}$ is half of unit ball B^3 with radius 2π



Group space of $SO(3)$ is doubly connected

Exercise 10: The Connection between $SU(2)$ and $SO(3)$

Consider $\mathcal{M} := \text{span}\,(\sigma_1, \sigma_2, \sigma_3)$, the vector space of traceless 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider mapping

$$M : \begin{array}{l} \mathbb{R}^3 \to \mathcal{M} \\ \vec{x} \mapsto M(\vec{x}) := \vec{\sigma} \cdot \vec{x} = \begin{pmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3 \end{pmatrix} \end{array}$$

Note $\det M(\vec{x}) = -|\vec{x}|^2$ and the mapping is bijective:

$$\begin{aligned}
\vec{x} &= \tfrac{1}{2}\text{Tr}\,(M(\vec{x})\vec{\sigma}) \\
&= \tfrac{\vec{e}_1}{2}\text{Tr}\,(M(\vec{x})\sigma_1) + \tfrac{\vec{e}_2}{2}\text{Tr}\,(M(\vec{x})\sigma_2) + \tfrac{\vec{e}_3}{2}\text{Tr}\,(M(\vec{x})\sigma_3) \\
&= \tfrac{\vec{e}_1}{2}\text{Tr}\begin{pmatrix} x_1 - \mathrm{i}x_2 & x_3 \\ -x_3 & x_1 + \mathrm{i}x_2 \end{pmatrix} + \tfrac{\vec{e}_2}{2}\text{Tr}\begin{pmatrix} \mathrm{i}x_1 + x_2 & -\mathrm{i}x_3 \\ -\mathrm{i}x_3 & -\mathrm{i}x_1 + x_2 \end{pmatrix} + \tfrac{\vec{e}_3}{2}\text{Tr}\begin{pmatrix} x_3 & -x_1 + \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_3 \end{pmatrix} \\
&= x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3
\end{aligned}$$

Homomorphism $SU(2) \to SO(3)$:

$$H : \begin{array}{l} SU(2) \to SO(3) \\ g \mapsto R(g) \end{array}, \qquad \text{where} \qquad M(R(g)\vec{x}) := gM(\vec{x})g^{-1}$$

Note $\det M(R(g))\vec{x} = \det M(\vec{x}) \Rightarrow |R(g)\vec{x}|^2 = |\vec{x}|^2$. That is $R(g)$ is rotation in $\mathbb{R}^3$.

Let $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $M(R(\pm E)\vec{x}) = (\pm E)M(\vec{x})(\pm E) = M(\vec{x})$.

Kernel of H: $Z_2 := \{E, -E\} \subset SU(2)$

Z_2 is center of $SU(2)$: Center of G is set $Z = \{z \in G | zg = gz\}$, is normal subgoup

$$SO(3) \simeq SU(2)/Z_2$$

Comments:

- Any multiple connected group has a unique universal covering group
  see, e.g., B. Wybourn "Classical Groups for Physicists"
- $SO(3)$ is doubly connected $\Rightarrow$ has single and double-valued reps
  $\ell = 0, 1, 2, \ldots$ single valued
  $\ell = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ double valued (= single valued for $SU(2)$)
  see, e.g., Joshi "Elements of Group Theory for Physicists"

Representations of $SO(3)$ in $\mathbb{R}^3$:
Rodriguez representation:

$$R(\vec{\omega}) := \mathbf{1}\cos\omega + (1 - \cos\omega)\vec{n}\vec{n}^T + \sin\omega \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \qquad \vec{\omega} = \omega\vec{n}$$

or with

$$N := \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad R(\omega) = \mathbf{1} + \sin\omega N + (1 - \cos\omega)N^2 \qquad \text{Proof!}$$

Example: $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow R(\vec{\omega}) = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Euler representation:

$$R(\varphi,\theta,\psi) = R_z(\varphi)R_x(\theta)R_z(\psi)$$

Rotations about x, y and z axis:

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, R_y(\alpha) := \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ \sin\alpha & 0 & \cos\alpha \\ 0 & 0 & 1 \end{pmatrix}, R_z(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Representations of $SO(3)$ in $L^2(\mathbb{R}^3)$:

$$\left(D(\vec{\omega})\psi\right)(\vec{x}) := \psi(R^{-1}(\vec{\omega})\vec{x}), \qquad \text{where} \qquad D(\vec{\omega}) = \exp\{-\mathrm{i}\vec{\omega}\cdot\vec{L}/\hbar\}$$

Angular momentum operator $\vec{L} = \vec{Q}\times\vec{P}$, $[L_i, L_j] = \mathrm{i}\hbar\varepsilon_{ijk}L_k$
Group law: $D(\omega\vec{n})D(\omega'\vec{n}) = D((\omega+\omega')\vec{n})$ for fixed axis
Unitarity: $D^{-1}(\vec{\omega}) = D(-\vec{\omega}) = D^\dagger(\vec{\omega})$
Decomposition of Hilbert space: $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^+)\otimes L^2(S^2)$ with $L^2(S^2) = \bigoplus_{\ell=0}^{\infty}\mathcal{D}^\ell$
$\mathcal{D}^\ell = \operatorname{span}\{|\ell m\rangle | m = -\ell,\ldots,\ell\}$, $\dim\mathcal{D}^\ell = 2\ell+1$

Representations of $SO(3)$ in Spin space $\mathbb{C}^{2s+1}$:
$\mathbb{C}^{2s+1} = \operatorname{span}\{|\vec{e},m\rangle | m = -s,\ldots,s\}$, $s = 0,\frac{1}{2},1,\frac{3}{2},\ldots$.
Quantisation axis $\vec{e}$: $\vec{S}\cdot\vec{e}|\vec{e},m\rangle = m\hbar|\vec{e},m\rangle$, usually $\vec{e} = \vec{e}_z$
Spin operator: $\vec{S}^2 = s(s+1)\hbar^2$
UIR: $D(\vec{\omega}) = \exp\{-\mathrm{i}\vec{\omega}\cdot\vec{S}/\hbar\}$
Rotation of quantisation axis:

$$D(\vec{\omega})(\vec{e}\cdot\vec{S})D^\dagger(\vec{\omega}) = (R(\omega)\vec{e})\cdot\vec{S}, \qquad D(\vec{\omega})|\vec{e},m\rangle = |R(\vec{\omega})\vec{e},m\rangle$$

ADDENDUM: $\hbar = 1$
Proof of $D(\vec{\omega}) = \exp\{-\mathrm{i}\vec{\omega}\cdot\vec{L}\} = \mathrm{e}^{-\mathrm{i}\varphi L_3}\mathrm{e}^{-\mathrm{i}\theta L_2}\mathrm{e}^{-\mathrm{i}\omega L_3}\mathrm{e}^{\mathrm{i}\theta L_2}\mathrm{e}^{\mathrm{i}\varphi L_3}$, $\qquad \vec{\omega} = \omega\vec{n}(\theta,\varphi)$
• Note: $L_3' := \mathrm{e}^{-\mathrm{i}\theta L_2}L_3\mathrm{e}^{\mathrm{i}\theta L_2} \Rightarrow \mathrm{e}^{-\mathrm{i}\omega L_3'} = \mathrm{e}^{-\mathrm{i}\theta L_2}\mathrm{e}^{-\mathrm{i}\omega L_3}\mathrm{e}^{\mathrm{i}\theta L_2}$ (use $A\mathrm{e}^B A^{-1} = \mathrm{e}^{ABA^{-1}}$)
Consider:

$$\frac{\partial L_3'}{\partial\theta} = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta L_2}\left(L_2L_3 - L_3L_2\right)\mathrm{e}^{\mathrm{i}\theta L_2} = \mathrm{e}^{-\mathrm{i}\theta L_2}L_1\mathrm{e}^{\mathrm{i}\theta L_2} \longrightarrow L_1 \quad \text{for } \theta\to 0$$

$$\frac{\partial^2 L_3'}{\partial\theta^2} = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta L_2}\left(L_2L_1 - L_1L_2\right)\mathrm{e}^{\mathrm{i}\theta L_2} = -\mathrm{e}^{-\mathrm{i}\theta L_2}L_3\mathrm{e}^{\mathrm{i}\theta L_2} = -L_3'$$

2. order differential eq. with initial conditions $L_3'|_{\theta=0} = L_3$ and $\left.\frac{\partial L_3'}{\partial\theta}\right|_{\theta=0} = L_1$
Integration yields: $L_3' = L_3\cos\theta + L_1\sin\theta$

Hence

$$\begin{aligned} D(\vec{\omega}) &= \mathrm{e}^{-\mathrm{i}\varphi L_3}\mathrm{e}^{-\mathrm{i}\omega L_3'}\mathrm{e}^{\mathrm{i}\varphi L_3} = \exp\{-\mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\varphi L_3}(L_3\cos\theta + L_1\sin\theta)\mathrm{e}^{\mathrm{i}\varphi L_3}\} \\ &= \exp\{-\mathrm{i}\omega(L_3\cos\theta + L_1'\sin\theta)\} \\ &\qquad \text{with} \qquad L_1' := \mathrm{e}^{-\mathrm{i}\varphi L_3}L_1\mathrm{e}^{\mathrm{i}\varphi L_3} = L_1\cos\varphi + L_2\sin\varphi \\ &= \exp\{-\mathrm{i}\omega(L_3\cos\theta + L_1\sin\theta\cos\varphi + L_2\sin\theta\sin\varphi)\} = \exp\{-\mathrm{i}\omega\vec{n}\cdot\vec{L}\} \end{aligned}$$

Transformation of vector operators

$$\exp\{-\mathrm{i}\vec{\omega}\cdot\vec{L}\}\vec{V}\exp\{\mathrm{i}\vec{\omega}\cdot\vec{L}\} = R^{-1}(\vec{\omega})\vec{V}$$

Exercise 10: Rotations in $\mathbb{R}^n$ and the Group $SO(n)$

Good reference:
N.Ja. Vilenkin, *Special Functions and the Theory of Group Representations* (1968)

Let $\vec{x} \in \mathbb{R}^n$ be represented in polar coordinates

$$\begin{aligned} x_1 &= r \sin\vartheta_{n-1} \sin\vartheta_{n-2} \cdots \sin\vartheta_2 \sin\vartheta_1 \\ x_2 &= r \sin\vartheta_{n-1} \sin\vartheta_{n-2} \cdots \sin\vartheta_2 \cos\vartheta_1 \\ x_3 &= r \sin\vartheta_{n-1} \sin\vartheta_{n-2} \cdots \cos\vartheta_2 \\ &\vdots \\ x_{n-1} &= r \sin\vartheta_{n-1} \cos\vartheta_{n-2} \\ x_n &= r \cos\vartheta_{n-1} \end{aligned} \qquad \text{where} \qquad \begin{aligned} &0 \le r < \infty \\ &0 \le \vartheta_k \le \pi \,, k = 2, 3, \ldots, n-1 \\ &0 \le \vartheta_1 < 2\pi \end{aligned}$$

Volume element:

$$\mathrm{d}^n\vec{x} = r^{n-1}\mathrm{d}r\, \mathrm{d}^{n-1}\Omega\,, \qquad \mathrm{d}^{n-1}\Omega = \sin^{n-2}\vartheta_{n-1} \sin^{n-3}\vartheta_{n-2} \cdots \sin\vartheta_2 \mathrm{d}\vartheta_{n-1} \cdots \mathrm{d}\vartheta_1$$

Volume:

$$\int_{S^{n-1}} \mathrm{d}^{n-1}\Omega = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Parametrisation of group elements
Let $g \in SO(n)$ such that $|g\vec{x}| = |\vec{x}|$ and $\det(g) = +1$, i.e. represent rotation matrix in $\mathbb{R}^n$
Let $g_k(\alpha)$ be (clockwise) rotation in plane $(x_k - x_{k+1})$ by angle α
Then each $g \in SO(n)$ can be represented by

$$g = g^{(n-1)} g^{(n-2)} \ldots g^{(1)} \qquad \text{where} \qquad g^{(k)} := g_1(\theta_1^k) g_2(\theta_2^k) \cdots g_k(\theta_k^k)$$
$$0 \le \theta_1^k < 2\pi\,, \qquad 0 \le \theta_j^k \le \pi\,, \quad j = 2, 3, \ldots, k$$

Obviously $g^{(k)} \in SO(k+1) \subset SO(n)$

Examples:
$n = 2$:

$$g = g_1(\theta_1^1) = \begin{pmatrix} \cos\theta_1^1 & \sin\theta_1^1 \\ -\sin\theta_1^1 & \cos\theta_1^1 \end{pmatrix}$$

$n = 3$:

$$\begin{aligned} g &= g^{(2)} g^{(1)} = g_1(\theta_1^2) g_2(\theta_2^2) g_1(\theta_1^1) \\ &= \begin{pmatrix} \cos\theta_1^2 & \sin\theta_1^2 & 0 \\ -\sin\theta_1^2 & \cos\theta_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2^2 & \sin\theta_2^2 \\ 0 & -\sin\theta_2^2 & \cos\theta_2^2 \end{pmatrix} \begin{pmatrix} \cos\theta_1^1 & \sin\theta_1^1 & 0 \\ -\sin\theta_1^1 & \cos\theta_1^1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Note: Number of group elements $\displaystyle\sum_{k=1}^{n-1}\sum_{j=1}^{k} 1 = \sum_{k=1}^{n-1} k = \frac{n}{2}(n-1) \Rightarrow \dim SO(n) = n(n-1)/2$

Consider: $h = g^{(n-2)} \cdots g^{(1)} \in SO(n-1) \subset SO(n)$ that is

$$h = \left(\begin{array}{c|c} SO(n-1) & 0 \\ \hline 0 & 1 \end{array} \right)$$

Let $\vec{n} = (0, 0, \ldots, 0, 1)^T$ be vector to "north pole", then $h\vec{n} = \vec{n}$ is invariant under $SO(n-1)$
$\Rightarrow$

$$\begin{aligned} g\vec{n} = g^{(n-1)}\vec{n} &= g_1(\theta_1^{n-1}) \cdots g_{n-1}(\theta_{n-1}^{n-1})\vec{n} \\ &= g_1(\theta_1^{n-1}) \cdots g_{n-2}(\theta_{n-2}^{n-1}) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sin\theta_{n-1}^{n-1} \\ \cos\theta_{n-1}^{n-1} \end{pmatrix} \\ &\vdots \\ &= \begin{pmatrix} \sin\theta_{n-1}^{n-1} \sin\theta_{n-2}^{n-1} \cdots \sin\theta_1^{n-1} \\ \sin\theta_{n-1}^{n-1} \sin\theta_{n-2}^{n-1} \cdots \cos\theta_1^{n-1} \\ \vdots \\ \sin\theta_{n-1}^{n-1} \cos\theta_{n-2}^{n-1} \\ \cos\theta_{n-1}^{n-1} \end{pmatrix} \end{aligned}$$

Obviously $\vec{x}/r = \vec{e}_x = g\vec{n}$ with $\vartheta_k = \theta_k^{(n-1)}$

Invariant Haar measure:

$$\mathrm{d}g = \prod_{l=2}^{n} \frac{\Gamma(l/2)}{2\pi^{l/2}} \prod_{k=1}^{n-1} \left(\prod_{j=1}^{k} \sin^{j-1}\theta_j^k \mathrm{d}\theta_j^k \right)$$

$$\mathrm{d}g = \mathrm{d}h \underbrace{\frac{\Gamma(n/2)}{2\pi^{n/2}} \mathrm{d}^{n-1}\Omega}_{\mathrm{d}\Gamma}$$

$\mathrm{d}\Gamma$: normalised $SO(n)$-invariant measure on $S^{n-1} = SO(n)/SO(n-1)$, $\int_{S^{n-1}} \mathrm{d}\Gamma = 1$

$n = 2:$ $\quad \mathrm{d}g = \frac{1}{2\pi} \mathrm{d}\theta_1^1$

$n = 3:$ $\quad \mathrm{d}g = \frac{1}{8\pi^2} \mathrm{d}\theta_1^1 \, \mathrm{d}\theta_1^2 \sin\theta_2^2 \mathrm{d}\theta_2^2$

$n = 4:$ $\quad \mathrm{d}g = \frac{1}{16\pi^4} \mathrm{d}\theta_1^1 \, \mathrm{d}\theta_1^2 \sin\theta_2^2 \mathrm{d}\theta_2^2 \, \mathrm{d}\theta_1^3 \sin\theta_2^3 \mathrm{d}\theta_2^3 \sin^2\theta_3^3 \mathrm{d}\theta_3^3$

Representations of class 1
Let $\mathcal{H} = L^2(S^{n-1})$

$$\mathcal{H} = \sum_{\ell=0}^{\infty} \mathcal{H}^\ell , \qquad \dim \mathcal{H}^\ell = (2\ell + n - 2) \frac{(\ell + n - 3)!}{\ell!(n-2)!} , \qquad \mathcal{H}^\ell = \mathrm{span}\,\{|\ell M\rangle\} ,$$

$M = (m_1, m_2, \ldots, m_{n-2})$, $\quad \ell, m_1, \ldots, m_{n-3} \in \mathbb{N}_0$, $m_{n-2} \in \mathbb{Z}$, $\quad \ell \equiv m_0 \geq m_1 \geq \cdots \geq m_{n-3} \geq |m_{n-2}|$

Let $\vec{e}_x \cdot \vec{n} = \cos\theta_{n-1}^{n-1}$ then with $O = (0, 0, \ldots, 0)$ and $D_{OO}^\ell(g) = \langle \ell O | D^\ell(g) | \ell O \rangle$

$$D_{OO}^\ell(g) = \frac{\ell!\Gamma(n-2)}{\Gamma(n+\ell-2)} C_\ell^{(n-2)/2}(\cos\theta_{n-1}^{n-1}) = D_{OO}^\ell(h_1 g h_2) , \qquad \forall\, h_1, h_2 \in SO(n-1)$$

form a complete set of zonal spherical functions on S^{n-1}, C_ℓ^ν Gegenbauer polynomials.

Hyperspherical harmonics:

$$Y_{\ell M}(\vec{e}_x) := \sqrt{d_\ell}\, D^\ell_{MO}(g) = \sqrt{d_\ell}\, D^\ell_{MO}(gh)\,, \qquad \forall\, h \in SO(n-1)$$

form orthonormal set on $\mathcal{H}$

$$\int_{S^{n-1}} \mathrm{d}\Gamma\, Y_{\ell M}(\vec{e}_x) Y^*_{\ell' M'}(\vec{e}_x) = \delta_{\ell\ell'}\delta_{MM}$$

Explicit expression

$$Y_{\ell M}(\vec{e}_x) = A_{\ell M}\, \mathrm{e}^{\mathrm{i}m_{n-2}\vartheta_1} \prod_{k=0}^{n-3} \left[C^{m_{k+1}+(n-k-2)/2}_{m_k+m_{k+1}}(\cos\vartheta_{n-k-1}) \sin\vartheta_{n-k-1} \right]$$

$$A^2_{\ell M} = \frac{1}{\Gamma(n/2)} \prod_{k=0}^{n-3} \frac{2^{2m_{k+1}+n-k-4}(m_k - m_{k+1})!}{\sqrt{\pi}\Gamma(m_{k+1}+m_k+n-k-2)} (n-k-2+2m_k)\Gamma^2(m_{k+1}+(n-k-2)/2)$$

Check for $n = 3$ with $m_0 = \ell$ and $m_1 = m \in \{-\ell, \ldots, \ell\}$

Gegenbauer polynomials (or ultra-spherical polynomials):

- Hypergeometric series: $C^\alpha_n(z) = \frac{(2\alpha)_n}{n!}\, {}_2F_1\left(-n, 2\alpha+n; \alpha+\frac{1}{2}; \frac{1-z}{2}\right)$

  Pochhammer symbol: $(a)_n := a(a-1)(a-2)\cdots(a-n+1) = \frac{\Gamma(a)}{\Gamma(a-n)}$

  Hypergeom. function: ${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a_n)(b)_n}{(c)_n} \frac{z^n}{n!}$

- Jacobi polynomials: $C^\alpha_n(z) = \frac{(2\alpha)_n}{(\alpha+\frac{1}{2})_n} P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(z)$

- Generalization of Legendre polynomials: $P_n(z) = C_n^{\frac{1}{2}}(z)$ i.e. $\alpha = \frac{1}{2}$

- Expansion in $\mathbb{R}^n$: $\vec{x}, \vec{y} \in \mathbb{R}^n$ $\quad \vec{x}\cdot\vec{y} = xy\cos\theta$

$$\frac{1}{|\vec{x}-\vec{y}|^{n-2}} = \sum_{\ell=0}^{\infty} \frac{x^\ell}{y^{\ell+n-2}}\, C_\ell^{\frac{n-2}{2}}(\cos\theta)$$

*** End of Tutorial 3 ***