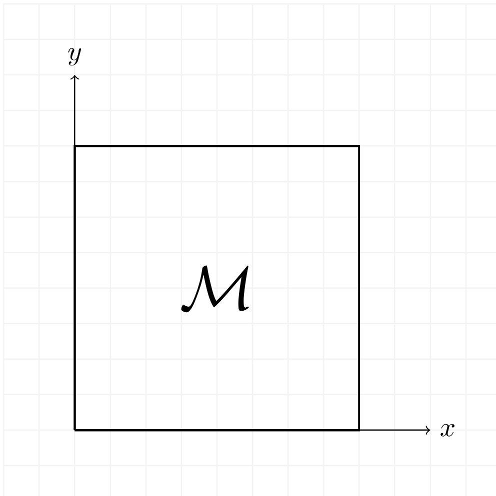
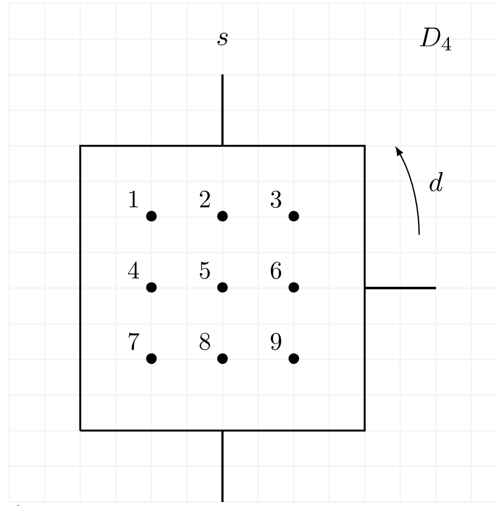
Exercise 4: Eigenmodes of a Membrane

Problem: Find the eigenmodes of a quadratic membrane approximated by 9 lattice sites





$\xi(x, y)$ obeys
$-\Delta\xi(x, y) = \omega^2\xi(x, y)$
with $\xi(\partial\mathcal{M}) = 0$

$\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_9) \in \mathbb{R}^9 =: V$
EV problem $M\vec{\xi} = \lambda\vec{\xi}$
M suitable 9×9 matrix

1) Choose proper Symmetry Group

$D_4 = \{e, d, d^2, d^3, s, sd, sd^2, sd^3\}$, $d^4 = e = s^2$, $sd = d^{-1}s = d^3s$

- Basis in V:

$$\vec{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \ldots \qquad \vec{e}_9 := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} =: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Representation of generators in V:

$$s \mapsto D^{(9)}(s) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & & 0 & & & 0 & \\ 1 & 0 & 0 & & & & & & \\ \hline & & & 0 & 0 & 1 & & & \\ & 0 & & 0 & 1 & 0 & & 0 & \\ & & & 1 & 0 & 0 & & & \\ \hline & & & & & & 0 & 0 & 1 \\ & 0 & & & 0 & & 0 & 1 & 0 \\ & & & & & & 1 & 0 & 0 \end{array}\right) \quad \text{as} \quad \begin{array}{c} 3 \to 1 \\ 2 \to 2 \\ 1 \to 3 \\ 6 \to 4 \\ 5 \to 5 \\ 4 \to 6 \\ 9 \to 7 \\ 8 \to 8 \\ 7 \to 9 \end{array}$$

$$d \mapsto D^{(9)}(d) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \quad \text{as} \quad \begin{array}{c} 3 \to 1 \\ 6 \to 2 \\ 9 \to 3 \\ 2 \to 4 \\ 5 \to 5 \\ 8 \to 6 \\ 1 \to 7 \\ 4 \to 8 \\ 7 \to 9 \end{array}$$

Let $\vec{\xi}_g := D^{(9)}(g)\vec{\xi}$ then $\vec{\xi}_g$ is also eigenvector of M to same eigenvalue λ:

$$MD^{(9)}(g) = D^{(9)}(g)M \qquad \text{for all} \qquad g \in D_4$$

M has D_4 symmetry

2) UIR of D_4

Without proof: D_4 has four 1-dim. and one 2-dim. UIR (see Stiefel & Fässler)
ord $D_4 = 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$ Burnside's theorem (later)

1-dim. UIR: Notation of Stiefel &Fässler

$$D^{(1,1)} : \begin{array}{l} d \mapsto 1 \\ s \mapsto 1 \end{array} \quad \text{trivial reps.} \qquad D^{(1,2)} : \begin{array}{l} d \mapsto 1 \\ s \mapsto -1 \end{array}$$

$$D^{(1,3)} : \begin{array}{l} d \mapsto -1 \\ s \mapsto 1 \end{array} \qquad D^{(1,4)} : \begin{array}{l} d \mapsto -1 \\ s \mapsto -1 \end{array}$$

2-dim. UIR:

$$D^{(2,1)} : \begin{array}{l} d \mapsto D^{(2,1)}(d) = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ s \mapsto D^{(2,1)}(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

3) Find Invariant Subspaces

$3 \times$ 1-dim. inv. subspaces belonging to $D^{(1,1)}$

$$\vec{x}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \vec{x}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

obviously $D^{(9)}(g)\vec{x}_i = \vec{x}_i$ for all $g \in D_4$

$1 \times$ 1-dim. inv. subspace belonging to $D^{(1,3)}$

$$\vec{y} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \left. \begin{array}{l} D^{(9)}(d)\vec{y} = -\vec{y} \\ \\ D^{(9)}(s)\vec{y} = \vec{y} \end{array} \right\} \quad \text{1-dim. reps } D^{(1,3)}(g)$$

$1 \times$ 1-dim. inv. subspace belonging to $D^{(1,4)}$

$$\vec{z} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad \left. \begin{array}{l} D^{(9)}(d)\vec{z} = -\vec{z} \\ \\ D^{(9)}(s)\vec{z} = -\vec{z} \end{array} \right\} \quad \text{1-dim. reps } D^{(1,4)}(g)$$

$2 \times$ 2-dim. inv. subspaces belonging to $D^{(1,2)}$

$$\vec{u}_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\left. \begin{array}{l} D^{(9)}(d)\vec{u}_1 = \vec{u}_2 \\ D^{(9)}(d)\vec{u}_2 = -\vec{u}_1 \end{array} \right\} D^{(9)}(d)\Big|_{\vec{u}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \left. \begin{array}{l} D^{(9)}(s)\vec{u}_1 = -\vec{u}_1 \\ D^{(9)}(s)\vec{u}_2 = \vec{u}_2 \end{array} \right\} D^{(9)}(s)\Big|_{\vec{u}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

$$\left.\begin{array}{l} D^{(9)}(d)\vec{v}_1 = \vec{v}_2 \\ D^{(9)}(d)\vec{v}_2 = -\vec{v}_1 \end{array}\right\} D^{(9)}(d)\Big|_{\vec{v}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \left.\begin{array}{l} D^{(9)}(s)\vec{v}_1 = -\vec{v}_1 \\ D^{(9)}(s)\vec{v}_2 = \vec{v}_2 \end{array}\right\} D^{(9)}(s)\Big|_{\vec{v}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conclusion:
$D^{(9)}(g) = 3 \cdot D^{(1,1)}(g) + 1 \cdot D^{(1,3)}(g) + 1 \cdot D^{(1,4)}(g) + 2 \cdot D^{(2,1)}(g)$
$D^{(9)}$ becomes block-diagonal under change of basis

$$\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_9,\} \qquad \Rightarrow \qquad \{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{y}, \vec{z}, \vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2\}$$

4) Constructing Operator M

Remember: $\dfrac{\mathrm{d}\xi(x,y)}{\mathrm{d}x} \Rightarrow \dfrac{\xi_i - \xi_{i-1}}{a} =: \dfrac{\Delta\xi_i}{a}$ and $\dfrac{\mathrm{d}^2\xi(x,y)}{\mathrm{d}x^2} \Rightarrow \dfrac{\Delta\xi_{i+1} - \Delta\xi_{i-1}}{a^2} = \dfrac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{a^2}$

Hence, with $a = 1$ we replace $-\frac{\partial^2\xi}{\partial x^2} \Rightarrow M_{xx}\xi_i = -\xi_{i-1} + 2\xi_i - \xi_{i+1}$, where $\xi_{i\pm1}$ is the excitation to left/right of point i. If there is no left/right this is set to zero (boundary condition). Similarly, we replace $-\frac{\partial^2\xi}{\partial y^2} \Rightarrow M_{yy}\xi_i = -\xi_{i-3} + 2\xi_i - \xi_{i+3}$, where $\xi_{i\pm3}$ is the excitation to above/below of point i. If there is no above/below this is set to zero (boundary condition).

The matrix $M = M_{xx} + M_{yy}$ is therefore represented by

$M\vec{e}_1 = 4\vec{e}_1 - \vec{e}_2 - \vec{e}_4$	only right and below neighbor
$M\vec{e}_2 = -\vec{e}_1 + 4\vec{e}_2 - \vec{e}_3 - \vec{e}_5$	only left/right and below
$M\vec{e}_3 = -\vec{e}_2 + 4\vec{e}_3 - \vec{e}_6$	only left and below
$M\vec{e}_4 = -\vec{e}_1 + 4\vec{e}_4 - \vec{e}_5 - \vec{e}_7$	only right and above/below
$M\vec{e}_5 = -\vec{e}_2 - \vec{e}_4 + 4\vec{e}_5 - \vec{e}_6 - \vec{e}_8$	all four neighbours
$M\vec{e}_6 = -\vec{e}_3 - \vec{e}_5 + 4\vec{e}_6 - \vec{e}_9$	only left and above/below
$M\vec{e}_7 = -\vec{e}_4 + 4\vec{e}_7 - \vec{e}_8$	only right and above
$M\vec{e}_8 = -\vec{e}_5 - \vec{e}_7 + 4\vec{e}_8 - \vec{e}_9$	only left and above/below
$M\vec{e}_9 = -\vec{e}_6 - \vec{e}_8 + 4\vec{e}_9$	only left and above

Hence M in old basis is given by

$$M = \left(\begin{array}{ccc|ccc|ccc} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{array}\right)$$

has D_4-symmetry as $MD^{(9)}(g) = D^{(9)}(g)M$ for all $g \in D_4$ (check for $g = \{d, s\}$!!!)

Construct now M in new basis

$$M\vec{x}_1 = M\vec{e}_5 = -\vec{e}_2 - \vec{e}_4 + 4\vec{e}_5 - \vec{e}_6 - \vec{e}_8 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix} = 4\vec{x}_1 - \vec{x}_2$$

$$M\vec{x}_2 = M(\vec{e}_2 + \vec{e}_4 + \vec{e}_6 + \vec{e}_8) = \begin{pmatrix} -2 & 4 & -2 \\ 4 & -4 & 4 \\ -2 & 4 & -2 \end{pmatrix} = -4\vec{x}_1 + 4\vec{x}_2 - 2\vec{x}_3$$

$$M\vec{x}_3 = M(\vec{e}_1+\vec{e}_3+\vec{e}_7+\vec{e}_9) = \begin{pmatrix} 4 & -2 & 4\\ -2 & 0 & -2 \\ 4 & -2 & 4\end{pmatrix} = -2\vec{x}_2 + 4\vec{x}_3$$

$\Sigma^{(1,1)} = \text{span}\,\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is the invariant subspace belonging to the UIR $D^{(1,1)}$

$$M\vec{y} = M(\vec{e}_2-\vec{e}_4-\vec{e}_6+\vec{e}_8) = \begin{pmatrix} 0 & 4 & 0\\ -4 & 0 & -4 \\ 0 & 4 & 0\end{pmatrix} = 4\vec{y} \qquad \Sigma^{(1,3)}$$

$$M\vec{z} = M(-\vec{e}_1+\vec{e}_3+\vec{e}_7-\vec{e}_9) = \begin{pmatrix} -4 & 0 & 4\\ 0 & 0 & 0 \\ 4 & 0 & -4\end{pmatrix} = 4\vec{z} \qquad \Sigma^{(1,4)}$$

$$M\vec{u}_1 = M(-\vec{e}_4+\vec{e}_6) = \begin{pmatrix} 1 & 0 & -1\\ -4 & 0 & 4 \\ 1 & 0 & -1\end{pmatrix} = 4\vec{u}_1-\vec{v}_1 \qquad M\vec{u}_2 = M(\vec{e}_2-\vec{e}_8) = \begin{pmatrix} -1 & 4 & -1\\ 0 & 0 & 0 \\ 1 & -4 & 1\end{pmatrix} = 4\vec{u}_2-\vec{v}_2$$

$$M\vec{v}_1 = M(-\vec{e}_1+\vec{e}_3-\vec{e}_7+\vec{e}_9) = \begin{pmatrix} -4 & 0 & 4\\ 2 & 0 & -2 \\ -4 & 0 & 4\end{pmatrix} = -2\vec{u}_1+4\vec{v}_1$$

$$M\vec{v}_2 = M(\vec{e}_1+\vec{e}_3-\vec{e}_7-\vec{e}_9) \begin{pmatrix} 4 & -2 & 4\\ 0 & 0 & 0 \\ -4 & 2 & -4\end{pmatrix} = -2\vec{u}_2+4\vec{v}_2$$

$\Sigma^{(2,1)} = \text{span}\,\{\vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2\}$ is the invariant subspace belonging to the UIR $D^{(2,1)}$

In new basis

$$M = \left(\begin{array}{ccc|c|c|cccc}
4 & -4 & 0 & & & & & & \\
-1 & 4 & -2 & & & & & & \\
0 & -2 & 4 & & & & & & \\
\hline
 & & & 4 & & & & & \\
\hline
 & & & & 4 & & & & \\
\hline
 & & & & & 4 & 0 & -2 & 0 \\
 & & & & & 0 & 4 & 0 & -2 \\
 & & & & & -1 & 0 & 4 & 0 \\
 & & & & & 0 & -1 & 0 & 4
\end{array}\right) \qquad \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline y \\ \hline z \\ \hline u_1 \\ u_2 \\ v_1 \\ v_2 \end{array}$$

Note that submatrix can be written as Kronecker product

$$\left(\begin{array}{cc|cc} 4 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 \\ \hline -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 4 \end{array}\right) = \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

change basis $\{u_1, u_2, v_1, v_2\} \Rightarrow \{u_1, v_1, u_2, v_2\}$ results in

$$M = \left(\begin{array}{ccc|c|c|cc|cc}
4 & -4 & 0 & & & & & & \\
-1 & 4 & -2 & & & & & & \\
0 & -2 & 4 & & & & & & \\
\hline
 & & & 4 & & & & & \\
\hline
 & & & & 4 & & & & \\
\hline
 & & & & & 4 & -2 & & \\
 & & & & & -1 & 4 & & \\
\hline
 & & & & & & & 4 & -2 \\
 & & & & & & & -1 & 4
\end{array}\right) \qquad \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \hline y \\ \hline z \\ \hline u_1 \\ v_1 \\ \hline u_2 \\ v_2 \end{array}$$

Eigenvalues and eigenvectors are now straightforward, at most 3×3 matrix problem, and are shown graphically on next page

$\Sigma^{(1,1)}$:

$$\begin{pmatrix} 4 & -4 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ \sqrt{2} \\ 1 \end{pmatrix} = (4-2\sqrt{2})\begin{pmatrix} 2 \\ \sqrt{2} \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 4 & -4 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 4\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 4 & -4 & 0 \\ -1 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ -\sqrt{2} \\ 1 \end{pmatrix} = (4+2\sqrt{2})\begin{pmatrix} 2 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$\Sigma^{(2,1)}$:

$$\begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} = (4-\sqrt{2})\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} = (4+\sqrt{2})\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

Eigenvalue	Eigenvector	Illustration
$4-2\sqrt{2}$	$2\vec{x}_1+\sqrt{2}\vec{x}_2+\vec{x}_3$	$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$
$4-\sqrt{2}$	$\sqrt{2}\vec{u}_1+\vec{v}_1$	$\begin{pmatrix} -1 & 0 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ -1 & 0 & 1 \end{pmatrix}$
$4-\sqrt{2}$	$\sqrt{2}\vec{u}_2+\vec{v}_2$	$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ 0 & 0 & 0 \\ -1 & -\sqrt{2} & -1 \end{pmatrix}$
4	$-2\vec{x}_1+\vec{x}_3$	$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}$
4	$\vec{y}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$
4	$\vec{z}$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$
$4+\sqrt{2}$	$\vec{v}_1-\sqrt{2}\vec{u}_1$	$\begin{pmatrix} 1 & 0 & -1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & 0 & -1 \end{pmatrix}$
$4+\sqrt{2}$	$\vec{v}_2-\sqrt{2}\vec{u}_2$	$\begin{pmatrix} -1 & \sqrt{2} & -1 \\ 0 & 0 & 0 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$
$4+2\sqrt{2}$	$2\vec{x}_1-\sqrt{2}\vec{x}_2+\vec{x}_3$	$\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$

Compare with exact result

$$\begin{aligned} \lambda_{mn} &= \frac{\pi^2}{a^2}(n^2+m^2) \qquad m,n=1,2,3,\dots \\ \xi_{mn}(x,y) &= N_{mn}\sin\left(m\frac{\pi x}{a}\right)\sin\left(n\frac{\pi y}{a}\right) \end{aligned}$$

Lowest mode

$$\begin{array}{ccc} 2N_{11}\xi_{11}\left(\frac{a}{4},\frac{3a}{4}\right)=1 & 2N_{11}\xi_{11}\left(\frac{a}{2},\frac{3a}{4}\right)=\sqrt{2} & 2N_{11}\xi_{11}\left(\frac{3a}{4},\frac{3a}{4}\right)=1 \\ 2N_{11}\xi_{11}\left(\frac{a}{4},\frac{a}{2}\right)=\sqrt{2} & 2N_{11}\xi_{11}\left(\frac{a}{2},\frac{a}{2}\right)=2 & 2N_{11}\xi_{11}\left(\frac{3a}{4},\frac{a}{2}\right)=\sqrt{2} \\ 2N_{11}\xi_{11}\left(\frac{a}{4},\frac{a}{4}\right)=1 & 2N_{11}\xi_{11}\left(\frac{a}{2},\frac{a}{4}\right)=\sqrt{2} & 2N_{11}\xi_{11}\left(\frac{3a}{4},\frac{a}{4}\right)=1 \end{array}$$

$$\Rightarrow \xi_{11} \sim \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} = 2\vec{x}_1 + \sqrt{2}\vec{x}_2 + \vec{x}_3$$

Quality of eigenvalues:

exact (degen.)		approx. (degen.)	
$\frac{\lambda_{12}}{\lambda_{11}} = \frac{5}{2} = 2,5$	(2)	$\frac{4-\sqrt{2}}{4-2\sqrt{2}} = 2,2$	(2)
$\frac{\lambda_{22}}{\lambda_{11}} = \frac{8}{2} = 4$	(1)	$\frac{4}{4-2\sqrt{2}} = 3,4$	(3)
$\frac{\lambda_{13}}{\lambda_{11}} = \frac{10}{2} = 5$	(2)	$\frac{4+\sqrt{2}}{4-2\sqrt{2}} = 4,6$	(2)

Exercise 5: Maschke's Theorem

Theorem: Each representation of a finite group is equivalent to a unitary representation with respect to scalar product

$$(x,y) := \sum_{i=1}^{d} x_i^* y_i \qquad x,y \in V\,, \qquad d = \dim V$$

Proof: Consider a non-unitary reps $D(g)$, that is, with $x^{(i)} := G(g_i)x$, $g_i \in G$ follows

$$(D(g_i)x, D(g_i)y) = (x^{(i)}, y^{(i)}) \neq (x,y)$$

Consider

$$\begin{aligned}\sum_{i=1}^{\operatorname{ord} G} (x^{(i)}, y^{(i)}) &= \sum_{g\in G} (D(g)x, D(g)y) \\ &= \sum_{g\in G} (\underbrace{D^\dagger(g)D(g)}_{\neq 1} x, y) =: (Hx, y) \qquad (1)\end{aligned}$$

with $H := \sum_{g\in G} D^\dagger(g)D(g) = H^\dagger \geq 0$

$\Rightarrow$ there exists a unitary U which diagonalize H (eigenvalue problem), that is

$$d := U^\dagger H U = \sum_g U^\dagger D^\dagger(g) U U^\dagger D(g) U =: \sum_g \tilde{D}^\dagger(g)\tilde{D}(g) \geq 0$$

As $d = d^\dagger \geq 0$ operator $d^{1/2}$ is well defined $\Rightarrow$

$$(Hx,y) = (UdU^\dagger x, y) = (dU^\dagger x, U^\dagger y) = (d^{1/2}U^\dagger x, d^{1/2}U^\dagger y) = (Lx, Ly) \qquad (2)$$

with $L := d^{1/2}U^\dagger$. Hence, $D'(g) := LD(g)L^{-1}$ is a unitary reps in the new basis $x' := Lx$ and $y' := Ly$.
Proof:

$$\begin{aligned}(D'(g)x', D'(g)y')' &= (LD(g)L^{-1}x', LD(g)L^{-1}y') \\ &= (LD(g)x, LD(g)y) \stackrel{(2)}{=} (HD(g)x, D(g)y) \\ &\stackrel{(1)}{=} \sum_{\tilde{g}} (D(\tilde{g})D(g)x, D(\tilde{g})D(g)y) \\ &= \sum_{\tilde{g}} (D(\tilde{g}g)x, D(\tilde{g}g)y) = \sum_{\tilde{g}} (D(\tilde{g})x, D(\tilde{g})y) \\ &\stackrel{(1)}{=} (Hx, y) \stackrel{(2)}{=} (Lx, Ly) = (x', y') \qquad \text{for all } g \in G\end{aligned}$$

Corollary: Each representation D of a finite group is unitary with respect to the scalar product

$$\langle x, y\rangle := \frac{1}{\operatorname{ord} G} \sum_{g\in G} (D(g)x, D(g)y)$$

Proof: For all $g_0 \in G$

$$\begin{aligned}\langle D(g_0)x, D(g_0)y\rangle &= \frac{1}{\operatorname{ord} G}\sum_{g\in G} (D(g)D(g_0)x, D(g)D(g_0)y) \\ &= \frac{1}{\operatorname{ord} G}\sum_{g\in G} (D(gg_0)x, D(gg_0)y) = \frac{1}{\operatorname{ord} G}\sum_{g\in G}(D(g)x, D(g)y) = \langle x, y\rangle\end{aligned}$$

Comment: Extension to compact and even uni-modular groups obvious by replacing

$$\frac{1}{\operatorname{ord} G}\sum_{g\in G}(\cdot) \qquad \text{by} \qquad \int_G \mathrm{d}g\,(\cdot)$$

Exercise 6: The UIRs of D_n

From homework (problem 6) we know that

$$
\begin{array}{lllll}
D_n & \text{has} & 4 + \left(\frac{n}{2} - 1\right) & \text{classes for} & n \quad \text{even} \\
D_n & \text{has} & 2 + \left(\frac{n-1}{2}\right) & \text{classes for} & n \quad \text{odd}
\end{array}
$$

From Burnside we know

$$
\operatorname{ord} D_n = 2n = \sum_j d_j^2 = \begin{cases} 4 \cdot 1^2 + \left(\frac{n}{2} - 1\right) \cdot 2^2 & n \quad \text{even} \\ 2 \cdot 1^2 + \left(\frac{n-1}{2}\right) \cdot 2^2 & n \quad \text{odd} \end{cases}
$$

So we expect for
n even: 4 1-dim. and $(n/2 - 1)$ 2-dim. UIR
n odd: 2 1-dim. and $(n-1)/2$ 2-dim. UIR

From exercise 4 we potentially know four 1-dim. UIR

1-dimensional UIRs

n even:

$$
\begin{array}{ll}
D^{(1,1)}(d) = 1\,, & D^{(1,1)}(s) = 1 \\
D^{(1,2)}(d) = 1\,, & D^{(1,2)}(s) = -1 \\
D^{(1,3)}(d) = -1\,, & D^{(1,3)}(s) = 1 \\
D^{(1,4)}(d) = -1\,, & D^{(1,4)}(s) = -1
\end{array}
$$

n odd:

$$
\begin{array}{ll}
D^{(1,1)}(d) = 1\,, & D^{(1,1)}(s) = 1 \\
D^{(1,2)}(d) = 1\,, & D^{(1,2)}(s) = -1
\end{array}
$$

Obviously the last two of case n even cannot be reps for n odd as $d^n = e$

2-dimensional UIRs

n **even:** Ansatz

$$
\tilde{D}^{(2,r)}(d) = \begin{pmatrix} \cos\alpha_r & -\sin\alpha_r \\ \sin\alpha_r & \cos\alpha_r \end{pmatrix} \qquad \text{with} \qquad \alpha_r = 2\pi\frac{r}{n} \qquad \text{as} \qquad d^n = e
$$

Note that rotation by φ and $2\pi - \varphi$ are equivalent due to $s \Rightarrow \alpha_r < \pi$
$\Rightarrow r = 1, 2, 3, \ldots, n/2 - 1$

Recall also

$$
\tilde{D}^{(2,r)}(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Equivalent reps:

$$
D^{(2,r)}(d) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \tilde{D}^{(2,r)}(d) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}r/n} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}r/n} \end{pmatrix}
$$

$$
D^{(2,r)}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \tilde{D}^{(2,r)}(s) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}
$$

n **odd:** Same as above but now $r = 1, 2, 3, \ldots, (n-1)/2$ as $\alpha_r = 2\pi r/n < \pi$.

Comment: In literature often

$$
D^{(2,r)}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

*** End of Tutorial 2 ***