Some calculations related to this

$$R = [\,Q_1^2 + Q_2^2 + \dots + Q_d^2\,]^{1/2} \qquad r = [\,x_1^2 + \dots + x_d^2\,]^{1/2}$$

$$[\,P_k, Q_l\,] = -i\,\delta_{kl}$$

$$[\,P_k, R\,] = -i\left(\frac{\partial r}{\partial x_k}\right) = -i\,\frac{x_k}{r} = -i\,\frac{Q_k}{R}$$

$$[\,Q_k, \vec{P}^2\,] = 2i\,P_k$$

$$\vec{Q}\cdot\vec{P} = x_k P_k = P_k x_k - [P_k, x_k] = \vec{P}\cdot\vec{Q} + i d$$

• Algebra: $J_1 = \frac{1}{2}(R\vec{P}^2 - R)$, $J_2 = \vec{Q}\cdot\vec{P} - \frac{d-1}{2}i$, $J_3 = \frac{1}{2}(R\vec{P}^2 + R)$

$$\begin{aligned}
[J_1, J_2] &= \tfrac{1}{2}\left[R\vec{P}^2 - R, \vec{Q}\cdot\vec{P} - \tfrac{d-1}{2}i\right] = \tfrac{1}{2}[R\vec{P}^2, \vec{Q}\cdot\vec{P}] - \tfrac{1}{2}[R, \vec{Q}\cdot\vec{P}] \\
&= \tfrac{1}{2}[R\vec{P}^2, Q_l P_l] - \tfrac{1}{2}[R, Q_k P_k] \\
&= \tfrac{1}{2}[R, Q_l P_l]\vec{P}^2 + \tfrac{1}{2}R[\vec{P}^2, Q_l P_l] - \tfrac{1}{2}Q_k[R, P_k] \\
&= \tfrac{1}{2}Q_l\left(i\tfrac{Q_l}{R}\right)\vec{P}^2 + \tfrac{1}{2}R(-2iP_l)P_l - \tfrac{1}{2}Q_k\left(i\tfrac{Q_k}{R}\right) \\
&= \tfrac{i}{2}R\vec{P}^2 - iR\vec{P}^2 - \tfrac{i}{2}R = -i\left(\tfrac{1}{2}R\vec{P}^2 + \tfrac{1}{2}R\right) = -iJ_3
\end{aligned}$$

$$\begin{aligned}
[J_2, J_3] &= \tfrac{1}{2}\left[\vec{Q}\cdot\vec{P} - \tfrac{d-1}{2}i, R\vec{P}^2 + R\right] = \tfrac{1}{2}[Q_k P_k, R\vec{P}^2] + \tfrac{1}{2}[Q_k P_k, R] \\
&= \tfrac{1}{2}Q_k[P_k, R\vec{P}^2] + \tfrac{1}{2}[Q_k, R\vec{P}^2]P_k + \tfrac{1}{2}Q_k\left(-i\tfrac{Q_k}{R}\right) \\
&= \tfrac{1}{2}Q_k\left(-i\tfrac{Q_k}{R}\right)\vec{P}^2 + \tfrac{1}{2}R(2iP_k)P_k - \tfrac{i}{2}R \\
&= -\tfrac{i}{2}R\vec{P}^2 + iR\vec{P}^2 - \tfrac{i}{2}R = i\left(\tfrac{1}{2}R\vec{P}^2 - \tfrac{1}{2}R\right) = iJ_1
\end{aligned}$$

$$\begin{aligned}
[J_3, J_1] &= \tfrac{1}{4}[R\vec{P}^2 + R, R\vec{P}^2 - R] = \tfrac{1}{4}[R, R\vec{P}^2] - \tfrac{1}{4}[R\vec{P}^2, R] = \tfrac{1}{2}R[R, \vec{P}^2] \\
&= \tfrac{1}{2}R[R, P_k]P_k + \tfrac{1}{2}RP_k[R, P_k] = \tfrac{1}{2}R\left(i\tfrac{Q_k}{R}\right)P_k + \tfrac{1}{2}RP_k\left(i\tfrac{Q_k}{R}\right) \\
&= \tfrac{i}{2}\vec{Q}\cdot\vec{P} + \tfrac{i}{2}RP_k\tfrac{Q_k}{R} = \tfrac{i}{2}\vec{Q}\cdot\vec{P} + \tfrac{i}{2}\left(P_k R + i\tfrac{Q_k}{R}\right)\tfrac{Q_k}{R} \\
&= \tfrac{i}{2}\vec{Q}\cdot\vec{P} + \tfrac{i}{2}\vec{P}\cdot\vec{Q} - \tfrac{1}{2} = \tfrac{i}{2}\left(\vec{Q}\cdot\vec{P} + \vec{Q}\cdot\vec{P} - id + i\right) = i\left(\vec{Q}\cdot\vec{P} - i\tfrac{d-1}{2}\right) = iJ_2
\end{aligned}$$

Casimir-Op:

$$\vec{J}^2 = -J_1^2 - J_2^2 + J_3^2 = -\frac{1}{4}\left(R\vec{P}^2 - R\right)^2 - \left(\vec{Q}\cdot\vec{P} - i\frac{d-1}{2}\right)^2 + \frac{1}{4}\left(R\vec{P}^2 + R\right)^2$$

$$= \frac{1}{4}\left[-\cancel{(R\vec{P}^2)^2} - \cancel{R^2} + R\vec{P}^2R + R^2\vec{P}^2 + \cancel{(R\vec{P}^2)^2} + \cancel{R^2} + R\vec{P}^2R + R^2\vec{P}^2\right]$$
$$- (\vec{Q}\cdot\vec{P})^2 + i(d-1)\,\vec{Q}\cdot\vec{P} + \left(\frac{d-1}{2}\right)^2$$

$$= \frac{1}{2}\left(R^2\vec{P}^2 + R\vec{P}^2R\right) - (\vec{Q}\cdot\vec{P})^2 + i(d-1)\,\vec{Q}\cdot\vec{P} + \left(\frac{d-1}{2}\right)^2$$

NR: $\vec{P}^2R = R\vec{P}^2 - [R, P_kP_k] = R\vec{P}^2 - P_k\left(i\frac{Q_k}{R}\right) - i\frac{Q_k}{R}P_k$

$$\rightsquigarrow \underline{\underline{R\vec{P}^2R}} = R^2\vec{P}^2 - i\,RP_k\frac{Q_k}{R} - i\vec{Q}\cdot\vec{P}$$
$$= R^2\vec{P}^2 - i\left(P_kR + i\frac{Q_k}{R}\right)\frac{Q_k}{R} - i\vec{Q}\cdot\vec{P} = R^2\vec{P}^2 - i\vec{P}\cdot\vec{Q} + 1 - i\vec{Q}\cdot\vec{P}$$
$$= R^2\vec{P}^2 + 1 - i\left(\vec{Q}\cdot\vec{P} - id\right) - i\vec{Q}\cdot\vec{P} = \underline{\underline{R^2\vec{P}^2 - 2i\vec{Q}\cdot\vec{P} - (d-1)}}$$

$$= R^2\vec{P}^2 - i\vec{Q}\cdot\vec{P} - \frac{d-1}{2} - (\vec{Q}\cdot\vec{P})^2 + i(d-1)\,\vec{Q}\cdot\vec{P} + \left(\frac{d-1}{2}\right)^2$$

$$= R^2\vec{P}^2 + i(d-2)\,\vec{Q}\cdot\vec{P} - (\vec{Q}\cdot\vec{P})^2 + \underbrace{\frac{d-1}{2}\left(\frac{d-1}{2} - 1\right)}_{= \frac{1}{4}(d-1)(d-3)}$$

Drehimpuls:

$$L_{ik} = Q_iP_k - Q_kP_i = -L_{ki}$$

$$\vec{L}^2 = \frac{1}{2}L_{ik}L_{ik} = \frac{1}{2}\left(Q_iP_k - Q_kP_i\right)\left(Q_iP_k - Q_kP_i\right)$$

$$= \frac{1}{2}\Big[Q_i\underbrace{P_kQ_i}_{Q_iP_k - i\delta_{ik}}P_k - Q_kP_i\underbrace{Q_iP_k}_{P_kQ_i + i\delta_{ik}} - Q_iP_k\underbrace{Q_kP_i}_{P_iQ_k + i\delta_{ik}} + Q_k\underbrace{P_iQ_k}_{Q_kP_i - i\delta_{ik}}P_i\Big]$$

$$= \frac{1}{2}\left[\vec{Q}^2\vec{P}^2 - i\vec{Q}\cdot\vec{P} - (\vec{Q}\cdot\vec{P})(\vec{P}\cdot\vec{Q}) - i\vec{Q}\cdot\vec{P} - (\vec{Q}\cdot\vec{P})(\vec{P}\cdot\vec{Q}) - i\vec{Q}\cdot\vec{P} + \vec{Q}^2\vec{P}^2 - i\vec{Q}\cdot\vec{P}\right]$$

$$= Q^2\vec{P}^2 - 2i\vec{Q}\cdot\vec{P} - (\vec{Q}\cdot\vec{P})\underbrace{(\vec{P}\cdot\vec{Q})}_{\vec{Q}\cdot\vec{P} - id}$$

$$= R^2\vec{P}^2 + i(d-2)\,\vec{Q}\cdot\vec{P} - (\vec{Q}\cdot\vec{P})^2$$

$$\Rightarrow \quad \vec{J}^2 = \vec{L}^2 + \frac{1}{4}(d-1)(d-3)$$

$$Q_i = \frac{1}{\sqrt{2}}\left(a_i^\dagger + a_i\right) \qquad P_i = \frac{i}{\sqrt{2}}\left(a_i^\dagger - a_i\right)$$

$$Q_i^2 = \frac{1}{2}\left(a_i^{\dagger 2} + a_i^\dagger a_i + a_i a_i^\dagger + a_i^2\right) = \frac{1}{2}\left(a_i^{\dagger 2} + a_i^2\right) + a_i^\dagger a_i + \frac{1}{2}$$

$$P_i^2 = -\frac{1}{2}\left(a_i^{\dagger 2} - a_i^\dagger a_i - a_i a_i^\dagger + a_i^2\right) = -\frac{1}{2}\left(a_i^{\dagger 2} + a_i^2\right) + a_i^\dagger a_i + \frac{1}{2}$$

$$\vec{Q}^2 = \sum Q_i^2 = 2J_1 + 2J_3$$

$$\vec{P}^2 = \sum P_i^2 = -2J_1 + 2J_3$$

$$\vec{Q}^2 \vec{P}^2 = \left(2J_1 + 2J_3\right)\left(2J_3 - 2J_1\right) = 4J_1J_3 + 4J_3^2 - 4J_1^2 - 4J_3J_1$$

$$= 4J_3^2 - 4J_1^2 - 4\left[J_3, J_1\right]$$

$$= 4J_3^2 - 4J_1^2 - 4iJ_2$$

$$\vec{Q}\cdot\vec{P} = \frac{i}{2}\sum\left(a_i^\dagger + a_i\right)\left(a_i^\dagger - a_i\right) = \frac{i}{2}\sum\left(a_i^{\dagger 2} - a_i^2 + \underbrace{a_i a_i^\dagger - a_i^\dagger a_i}_{1}\right)$$

$$= -2J_2 + \frac{i}{2}d$$

$$\left(\vec{Q}\cdot\vec{P}\right)^2 = 4J_2^2 - i2dJ_2 - \frac{d^2}{4}$$

$$L^2 = \vec{Q}^2\vec{P}^2 + i(d-2)\,\vec{Q}\cdot\vec{P} - \left(\vec{Q}\cdot\vec{P}\right)^2 =$$

$$= 4J_3^2 - 4J_1^2 - 4iJ_2 + i(d-2)\left(\frac{i}{2}d - 2J_2\right) - 4J_2^2 + i2dJ_2 + \frac{d^2}{4}$$

$$= 4\vec{J}^2 + iJ_2\big(\underbrace{-4 - 2(d-2) + 2d}_{=0}\big) - \frac{d}{2}(d-2) + \frac{d^2}{4}$$

$$= 4\vec{J}^2 - \frac{d^2}{4} + d = 4\vec{J}^2 - \frac{d}{4}(d-4)$$