3.5 Harmonic Analysis on Homogenous Spaces

Let $f(g) \in L^2(G/H)$, H massive with $H\vec{x}_0 = \vec{x}_0$
$\Rightarrow f(\vec{x}) = f(g\vec{x}_0) =: \hat{f}(g) \in L^2(G)$ such that $\hat{f}(gh) = \hat{f}(g)$ for all $h \in H$.
All $f(g) \in L^2(G/H)$ are associate spherical functions.

Peter-Weyl Theorem: continuous version

$$\boxed{\begin{aligned} \hat{f}(g) &= \sum_{\text{all UIR } j \text{ of } G} d_j \sum_{m,n} \tilde{f}^j_{nm}\, D^j_{mn}(g) \\ \tilde{f}^j_{nm} &= \int_G \mathrm{d}g\, \hat{f}(g)\, D^{j\,*}_{mn}(g) \end{aligned}}$$

Fourier coefficient for associate spherical functions:

$$\tilde{f}^j_{nm} = \int_G \mathrm{d}g\, \hat{f}(g)\, D^{j\,*}_{mn}(g) = \int_G \mathrm{d}g\, \hat{f}(g)\, D^{j\,*}_{mn}(gh) = \int_G \mathrm{d}g\, \hat{f}(g) \sum_k D^{j\,*}_{mk}(g) D^{j\,*}_{kn}(h)$$

Integration over h:

$$\tilde{f}^j_{nm} = \int_G \mathrm{d}g\, \hat{f}(g)\, D^{j\,*}_{mn}(g) = \int_G \mathrm{d}g\, \hat{f}(g)\, D^{j\,*}_{mn}(gh) = \int_G \mathrm{d}g\, \hat{f}(g) \sum_k D^{j\,*}_{mk}(g) \int_H \mathrm{d}h\, D^{j\,*}_{kn}(h)$$

Note: The UIR D^j of G is in general a reducible unitary reps of $H \subset G$. So let's decompose into UIR D^α of H

$$D^j = \sum_\alpha c_\alpha\, D^\alpha$$

1) Let $j \neq$ class 1: No invariant vector in $D^j \Rightarrow D^j$ does not contain trivial representation

$$\int_H \mathrm{d}h\, D^{j\,*}_{kn}(h) = 0 \qquad \text{for all} \qquad j \neq \text{class } 1$$

2) Let $j =$ class 1: As H is massive $\Rightarrow$ exists only one vector $|\varphi_0\rangle$ such that $D^j(h)|\varphi_0\rangle = |\varphi_0\rangle$.
That is, $D^j_{00}(h) = \langle\varphi_0|D^j(h)|\varphi_0\rangle = 1$.
In other words, the trivial reps of H appears exactly once in above decomposition and

$$\int_H \mathrm{d}h\, D^{j\,*}_{kn}(h) = \delta_{k0}\delta_{n0} \qquad \text{for all} \qquad j = \text{class } 1$$

Result:

$$\tilde{f}^j_{nm} = \int_G \mathrm{d}g\, \hat{f}(g)\, D^{j\,*}_{m0}(g)\delta_{n0}$$

Let Λ be the set of class 1 reps of G:

$$\begin{aligned} f(\vec{x}) = \hat{f}(g) &= \sum_{j\in\Lambda} d_j \sum_{m=0}^{d_j-1} \tilde{f}^j_m\, D^j_{m0}(g) \\ \tilde{f}^j_m &= \int_G \mathrm{d}g\, \hat{f}(g)\, D^{j\,*}_{m0}(g) \end{aligned}$$

Generalised spherical harmonics:

$$Y_{\ell m}(\vec{x}) := \sqrt{d_\ell}\, D^\ell_{m0}(g)\,, \qquad \vec{x} = g\vec{x}_0$$

Form complete orthonormal set on $L^2(G/H)$:

$$\int_{G/H} \mathrm{d}\mu(\vec{x})\, Y_{\ell m}(\vec{x}) Y^*_{\ell' m'}(\vec{x}) = \int_G \mathrm{d}g\, \sqrt{d_\ell d_{\ell'}}\, \underbrace{D^\ell_{m0}(g) D^{\ell'\,*}_{m'0}(g)}_{\delta'_{\ell\ell}\delta_{mm'}/d_\ell} = \delta'_{\ell\ell}\delta_{mm'}$$

Harmonic analysis on homogenous spaces G/H (with massive H):

$$\boxed{\begin{aligned} f(\vec{x}) &= \sum_{\ell\in\Lambda}\sum_{m} c_{\ell m}\, Y_{\ell m}(\vec{x}) \\ c_{\ell m} &= \int_{G/H} \mathrm{d}\mu(\vec{x})\, f(\vec{x})\, Y^*_{\ell m}(\vec{x}) \end{aligned}}$$

Note: $Y_{\ell m}(\vec{x})$ are also eigenfunctions of Laplace-Beltrami operator on $\mathcal{M} = G/H$.

Harmonic analysis for zonal spherical functions: $f(h_1^{-1} g h_2) = f(g)$, $h_1, h_2 \in H$

$$\boxed{\begin{aligned} f(g) &= \sum_{\ell\in\Lambda} d_\ell\, \lambda_\ell\, D^\ell_{00}(g) \\ \lambda_\ell &= \int_G \mathrm{d}g\, f(g)\, D^{\ell\,*}_{00}(g) \end{aligned}}$$

3.6 Generators of Lie Groups

Let G be a Lie group with $g = g(\alpha) \in G$, $\alpha = (\alpha^1, \ldots, \alpha^n)$, $e = g(0)$ neutral element
Consider reps of G in some $\mathcal{H}$: $D(g) = D(g(\alpha))$ with $D(g(0)) = 1$

Generator:

$$X_a := \left.\frac{\partial D(g(\alpha))}{\partial \alpha^a}\right|_{\alpha=0}$$

In quantum mechanics often $T_a = \pm \mathrm{i} X_a$ as they are self-adjoint operators for unitary reps

Generators are in essence the reps matrices near neutral element as

$$\begin{aligned} D(g(\delta\alpha)) &= 1 + \delta\alpha^a X_a + O(\delta\alpha^2) \\ D(g^{-1}(\delta\alpha)) &= [D(g(\delta\alpha))]^{-1} = 1 - \delta\alpha^a X_a + O(\delta\alpha^2) \end{aligned}$$

We now use summation convention, so in above sum over a

Comments:

- Generators depend on parametrisation of G
- For unitary reps $D(g^{-1}(\delta\alpha)) = D^\dagger(g(\delta\alpha)) \Rightarrow X_a^\dagger = -X_a \quad$ or $\quad T_a^\dagger = T^a$
- For $\dim \mathcal{H} = d < \infty$: X_a or T_a are $d \times d$ matrices
- For $\dim \mathcal{H} = \infty$: X_a or T_a are linear operators acting on vectors in $\mathcal{H}$

Consider:

$$\begin{aligned} D(g(\delta\gamma)) &= D(g^{-1}(\beta) g(\delta\alpha) g(\beta)) \\ &= 1 + \delta\alpha^a D(g^{-1}(\beta)) X_a D(g(\beta)) \\ &= 1 + \delta\gamma^a X_a \end{aligned}$$

That is $D(g^{-1}(\beta)) X_a D(g(\beta))$ is a linear combination of generators
Now $\beta \to \delta\beta$ small

$$(1 - \delta\beta^b X_b) X_a (1 + \delta\beta^b X_b) = X_a + \delta\beta^b (X_a X_b - X_b X_a) + O(\delta\beta^2)$$

$\Rightarrow$

$$\boxed{[X_a, X_b] := (X_a X_b - X_b X_a) = c^d_{ab} X_d}$$

The constants c^d_{ab} are called *structure constants*

Properties

- Depend on parametrisation of H
- $c^d_{ab} = -c^d_{ba}$ antisymmetric in lower indices, $c^d_{aa} = 0$
- From Jacobi identity $[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0$ follows

$$c^d_{ab} c^e_{dc} + c^d_{bc} c^e_{da} + c^d_{ca} c^e_{db} = 0$$

- G abelian $\Leftrightarrow c^d_{ab} = 0$
  Obvious as $D(g^{-1}(\beta)) X_a D(g(\beta)) = X_a \Leftrightarrow [X_a, X_b] = 0$ for all a, b, d

Example: $G = SO(3)$ (see Homework 4 Problem 10 with $\hbar = 1$)

$$[L_i, L_j] = \mathrm{i}\varepsilon_{ijk} L_k \qquad \Rightarrow \qquad c^k_{ij} = \mathrm{i}\varepsilon_{ijk}$$

3.7 Generators for transformation groups

Consider transfomation on $\mathcal{M}$: $x' = g(\alpha)x$ and let $\alpha \to \delta\alpha$
Then the μ-component of x'_μ is given by

$$x'_\mu = x_\mu + \delta x_\mu = x_\mu + \delta\alpha^a \, U_{a\mu}(x) + O(\delta^2\alpha)$$

where

$$U_{a\mu}(x) := \frac{\delta x_\mu}{\delta \alpha^a} \, .$$

Consider reps in $\mathcal{H} = L^2(\mathcal{M})$:

$$(D(g(\alpha))\psi)(x) = \psi(g^{-1}(\alpha)x)$$

then again let $\alpha \to \delta\alpha$

$$(\psi + \delta\alpha^a X_a \psi)(x) = \psi(x - \delta x) = \psi(x) - \delta\alpha^a U_{a\mu}(x) \frac{\partial \psi}{\partial x_\mu}(x)$$

Generator: Is a linear differential operator on $\mathcal{H}$

$$\boxed{X_a = -U_{a\mu}(x) \frac{\partial}{\partial x_\mu} = -U_{a\mu}(x)\partial^\mu}$$

Example: $\mathcal{M} = \mathbb{R}^2$, $G = SO(2)$

$$\begin{aligned} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} &= \begin{pmatrix} \cos(\delta\alpha) & -\sin(\delta\alpha) \\ \sin(\delta\alpha) & \cos(\delta\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} 1 & -\delta\alpha \\ \delta\alpha & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \delta\alpha \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \end{aligned}$$

Hence $U_1(x) = \dfrac{\delta x_1}{\delta\alpha} = -x_2$ and $U_2(x) = \dfrac{\delta x_2}{\delta\alpha} = x_1$
$\Rightarrow$

$$X = -U_\mu(x)\partial^\mu = -\left(-x_2\partial^1 + x_1\partial^2\right) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

Or with $T = \mathrm{i}\hbar X$

$$T = \mathrm{i}\hbar \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right) = L_3$$

Rotations in the plane are generated by the angular momentum operator L_3 on $L^2(\mathbb{R}^2)$.

Generator of rotation matrix (2-dim reps in $\mathbb{R}^2$)

$$\begin{pmatrix} \cos(\delta\alpha) & -\sin(\delta\alpha) \\ \sin(\delta\alpha) & \cos(\delta\alpha) \end{pmatrix} \approx 1 + \delta\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1 - \mathrm{i}\delta\alpha\,\sigma_2$$

Generator for rotation matrix in $\mathbb{R}^2$ is given by Pauli matrix $X = -\mathrm{i}\sigma_2$
Finite rotation by angle α is given

$$\begin{aligned} \mathrm{e}^{\alpha X} &= \sum_{n=0}^{\infty} \frac{1}{n!}\alpha^n X^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\mathrm{i}\alpha)^n \sigma_2^n \qquad (\sigma_2^2 = 1) \\ &= \mathbf{1}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\alpha^{2k} - \mathrm{i}\sigma_2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\alpha^{2k+1} \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\alpha\sigma_2} \end{aligned}$$

Finite group elements can be represented by generators via exponentiation.

For each generator exists an one-parameter subgroup represented by

$$D(g(t)) = \mathrm{e}^{tX_t}$$

Recall with $\delta t = t/n$ for large n

$$D(g(t)) = [D(g(\delta t))]^n = \lim_{n\to\infty} [1 + \delta t X_t]^n = \lim_{n\to\infty} \left[1 + \frac{t}{n}\, X_t\right]^n = \mathrm{e}^{tX_t}$$

4 Lie Algebras

4.1 Definitions

A finite-dimensional vector space $\mathcal{L}$ (real or complex) is called *Lie Algebra* if there exists a *Lie Product* $\circ$ such that

- $X \circ Y \in \mathcal{L} \qquad \forall\, X, Y \in \mathcal{L}$
- $(\alpha X + \beta Y) \circ Z = \alpha X \circ Y + \beta Y \circ Z \qquad \forall\, X, Y, Z \in \mathcal{L}\,, \qquad \alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$
- $Z \circ Y = -Y \circ X$ antisymmetric
- $X \circ (Y \circ Z) + Y \circ (Z \circ X) + Z \circ (X \circ Y) = 0$ Jacobi Identity

Example: $\mathcal{L} = \mathbb{R}^3$ with $\vec{x} \circ \vec{y} := \vec{x} \times \vec{y}$

Theorem: The generators of a Lie Group form a Lie Algebra with

$$X \circ Y := [X, Y] = XY - YX$$

Comments:

- Notation $G = SO(n) \Leftrightarrow \mathcal{L} = so(n)$, that is, use lower case notation for algebra
- *locally isomorphic* groups :$\Leftrightarrow$ have the same (isomorphic) Lie Algebra
  Example: $so(3) \simeq su(2)$ are isomorphic
- Reps of group $\Leftrightarrow$ Reps of algebra
- $\dim G = \dim \mathcal{L}$

From now on we consider only

$$X \circ Y = [X, Y]$$

Homomorphism:

$$T: \begin{array}{l} \mathcal{L} \to \mathcal{L}' \\ X \mapsto T(X) \end{array}$$

such that $T(\alpha X + \beta Y) = \alpha T(x) + \beta T(Y)$ and $T([X, Y]) = [T(X), T(Y)]$

Subalgebra: Subspace $\mathcal{N} \subset \mathcal{L}$ with $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$

Ideal: Subalgebra $\mathcal{N} \subset \mathcal{L}$ with $[\mathcal{L}, \mathcal{N}] \subset \mathcal{N}$
trivial ideals are $\mathcal{L} = \{0\}$ and $\mathcal{N} = \mathcal{L}$, rest are called proper ideals

Center: max. ideal such that $[\mathcal{N}, \mathcal{N}] = 0$ abelian algebra

Simple Lie algebra: $\mathcal{L}$ has no proper ideal

Semi-simple Lie Algebra: $\mathcal{L}$ has no abelian ideals

4.2 Representations of Lie Algebras

Homomorphism:

$$D: \begin{array}{l} \mathcal{L} \to \text{lin. operators on } \mathcal{D} \\ X \mapsto D(X) \end{array}$$

is called *representation* of $\mathcal{L}$ in $\mathcal{D}$

Reps of Lie Group	$\Leftrightarrow$	Reps of Lie Algebra
$D(g(\alpha)) = \mathrm{e}^{\alpha^a X_a}$		X_a
$D(g(\alpha)) = \mathrm{e}^{\mathrm{i}\alpha^a T_a}$		T_a
unitary		$X_a^\dagger = -X_a\,, T_a^\dagger = T_a$
irreducible		irreducible

Adjoint Representation:
Let $X \in \mathcal{L}$ be fixed

$$ad(X): \begin{array}{l} \mathcal{L} \to \mathcal{L} \\ Y \mapsto [X, Y] \end{array}$$

$$\begin{aligned} [ad(X), ad(Y)]Z &= ad(X)ad(Y)Z - ad(Y)ad(X)Z \\ &= ad(X)[Y, Z] - ad(Y)[X, Z] \\ &= [X, [Y, Z]] - [Y, [X, Z]] \\ &= -[Z, [X, Y]] \qquad \text{Jacobi Id.} \\ &= [[X, Y], Z] \\ &= ad([X, Y])Z \qquad \forall\, Z \in \mathcal{L} \end{aligned}$$

Let $X_1, \ldots, X_n$ be basis in $\mathcal{L}$ with

$$[X_i, X_k] = c_{ik}^l\, X_l$$

then

$$ad(X_i)X_k = [X_i, X_k] = c_{ik}^l\, X_l$$

The structure constants are the matrix elments of the adjoint representation

$$ad(X_i)_k^l = c_{ik}^l$$

Note: $\dim ad(X) = \dim \mathcal{L} = \dim G$

Example $so(3)$: $\dim so(3) = 3$ (see Homework 4 Problem 10)

4.3 Cartan Metric

Scalar product in $\mathcal{L}$: (Killing form)

$$(X, Y) := \mathrm{Tr}\,(ad(X)ad(Y))$$

Using above basis $X_1, \ldots, X_n$ in $\mathcal{L}$

Cartan metric:

$$g_{kl} := \mathrm{Tr}\,(ad(X_k)ad(X_l)) = c_{kr}^s c_{ls}^r = g_{lk} \qquad \text{symmetric}$$

Let $X = a^k X_k$ and $Y = b^l X_l$ then

$$(X, Y) = \mathrm{Tr}(a^k\, ad(X_k)\, b^l\, ad(X_l)) = \mathrm{Tr}(ad(X_k)\, ad(X_l))\, a^k b^l = g_{kl} a^k b^l$$

Cartan Criterium:

$$\det(g_{kl}) \neq 0 \qquad \Leftrightarrow \qquad \mathcal{L} \text{ semi-simple}$$

This implies the existence of the inverse metric tensor: $g_{kl} g^{lm} = \delta_k^l$

Lie algebra *compact* $:\Leftrightarrow$ g_{kl} positive or negative definite

Examples:

- $so(3)$: $\quad [X_i, X_j] = \varepsilon_{ijk} X_k \quad \Rightarrow g_{kl} = \varepsilon_{krs}\varepsilon_{lrs} = 2\delta_{kl}$
  semi-simple and compact
  also with $L_i = \mathrm{i}X_i \quad \Rightarrow \quad [L_i, L_j] = \mathrm{i}\varepsilon_{ijk} L_k \quad \Rightarrow g_{kl} = -2\delta_{kl}$

- $so(2,1)$: $\quad [X_1, X_2] = X_3,\ [X_2, X_3] = -X_1,\ [X_3, X_1] = X_2$

  $\Rightarrow \quad c^3_{12} = 1 = -c^3_{21}\,, \quad c^1_{23} = -1 = -c^1_{32}\,, \quad c^2_{31} = 1 = -c^2_{13}$

  $\Rightarrow \quad g_{11} = \mathrm{Tr}\,(ad(X_1)ad(X_1)) = 2c^3_{12}c^2_{13} = -2$
  $\Rightarrow \quad g_{22} = \mathrm{Tr}\,(ad(X_2)ad(X_2)) = 2c^1_{23}c^3_{21} = 2$
  $\Rightarrow \quad g_{33} = \mathrm{Tr}\,(ad(X_3)ad(X_3)) = 2c^2_{31}c^1_{32} = 2$

  $$\Rightarrow (g_{kl}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{semi-simple and NOT compact}$$

  Let $T_i = \mathrm{i}X_i \quad \Rightarrow \quad [T_1, T_2] = \mathrm{i}T_3,\ [T_2, T_3] = -\mathrm{i}T_1,\ [T_3, T_1] = -\mathrm{i}T_2$

  $$\Rightarrow (g_{kl}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

  This $so(2,1) \simeq su(1,1)$ algebra is often used in QM as *spectrum generating algebra*
  T_3 is called the compact operator

Consider:
$c_{ijk} := g_{il}c^l_{jk} = c^s_{ir}c^r_{ls}c^l_{jk} = c^s_{ir}\left(-c^l_{ks}c^r_{lj} - c^l_{sj}c^r_{lk}\right) = c^s_{ir}c^r_{jl}c^l_{ks} + c^s_{ri}c^l_{sj}c^r_{lk}$
$\Rightarrow c_{ijk}$ is totally anti-symmetric

4.4 The Casimir Operator

Definition:

$$\boxed{C := g^{kl} X_k X_l = g_{kl} X^k X^l}$$

Consider

$$\begin{aligned}
[C, X_i] &= g^{kl}[X_k X_l, X_i] \\
&= g^{kl} X_k [X_l, X_i] + g^{kl}[X_k, X_i] X_l \\
&= g^{kl} X_k\, c^r_{li} X_r + g^{kl}\, c^r_{ki} X_r X_l \\
&= g^{kl} X_k\, c^r_{li} X_r + g^{kl}\, c^r_{li} X_r X_k \qquad \text{as } g^{kl} = g^{lk} \\
&= g^{kl} c^r_{li}\, (X_k X_r + X_r X_k) \\
&= \underbrace{g^{kl} g^{rs} \underbrace{(X_k X_r + X_r X_k)}_{sym.\, k \leftrightarrow r}}_{sym.\, l \leftrightarrow s} \underbrace{c_{lis}}_{antisym.\, l \leftrightarrow s} = 0
\end{aligned}$$

The Casimir operator commutes with all elements of $\mathcal{L}$
In case of an irreducible reps follows via Schur lemma: $C = \lambda \mathbf{1}$
These eigenvalues are used to characterise the irreducible reps

Examples:

- $so(3) \simeq su(2)$: $g_{kl} = -2\delta_{kl}\,, \quad g^{kl} = -\frac{1}{2}\delta^{kl}$
  $C = -\frac{1}{2}\left(X_1^2 + X_2^2 + X_3^2\right) = \frac{1}{2}\vec{J}^2$ with $J_k = -\mathrm{i}X_k$, $\vec{J}^2 = j(j+1)\mathbf{1}$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

- $so(2,1) \simeq su(1,1)$:

  $$g_{kl} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad C = \frac{1}{2}(-X_1^2 + X_2^2 + X_3^2) \qquad \text{unbounded operator}$$

Comment: The UIR of non-compact groups/algebras are ALL infinite dimensional. This becomes a problem for Lorentz group $SO(3,1) = SO(3) \otimes SO(2,1)$ and the classification of elementary particles. Way out is to use non-unitary but finite-dimensional irreducible reps. More later (Wigner states).

Generalisation of Racah:

$$C_n := c_{k_1 l_1}^{l_2} c_{k_2 l_2}^{l_3} \cdots c_{k_n l_n}^{l_1} X^{k_1} X^{k_2} \cdots X^{k_n}$$

commute with all X_i: $[C_n, X_i] = 0$

Range of $\mathcal{L}$:⇔ No. of independent C_n's

4.5 Representations of Lie Algebras in Quantum Mechanics

4.5.1 The angular momentum algebra $so(3) \simeq su(2)$

Algebra:

$$[J_i, J_k] = \mathrm{i}\hbar\varepsilon_{ikl} J_l$$

UIR have dimension $d_j = 2j + 1$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$
$\mathcal{D}^j = \text{span}\{|j,m\rangle | m = -j, \dots, j\} \simeq \mathbb{C}^{2j+1}$

Cartan-Weyl basis: $J_\pm := J_1 \pm J_2$, $J_0 := J_3$

$$[J_0, J_\pm] = \pm\hbar J_\pm , \qquad [J_+, J_-] = 2\hbar J_0$$

Usual basis: $J_0|j,m\rangle = m\hbar|j,m\rangle \Rightarrow J_\pm$ change eigenvalue by $\pm\hbar$
Ansatz:

$$J_\pm|j,m\rangle = N_\pm\hbar|j, m \pm 1\rangle$$

Casimir: $C = \frac{1}{2}\vec{J}^2 \sim \mathbf{1} \Rightarrow \vec{J}^2|j,m\rangle = \lambda_j|j,m\rangle$ with $\lambda_j \geq 0$ as $\vec{J}^2 \geq 0$.

Calculation of $N_\pm$: $(J_\pm)^\dagger = J_\mp$, $J_0^\dagger = J_0$

$$\begin{aligned} |N_\pm|^2\hbar^2 &= ||J_\pm|j,m\rangle||^2 = \langle j,m|J_\mp J_\pm|j,m\rangle = \langle j,m|[\vec{J}^2 - J_0(J_0 \pm 1)]|j,m\rangle \\ &= [\lambda_j - m(m \pm 1)]\hbar^2 \geq 0 \end{aligned}$$

Hence m must be bounded $m_{min} \leq m \leq m_{max}$ with $J_+|j, m_{max}\rangle = 0$ and $J_-|j, m_{min}\rangle = 0$
Consider

$$\vec{J}^2|j, m_{max}\rangle = J_- J_+|j, m_{max}\rangle + J_0(J_0 + 1)|j, m_{max}\rangle = m_{max}(m_{max} + 1)|j, m_{max}\rangle$$

$$\vec{J}^2|j, m_{min}\rangle = J_+ J_-|j, m_{min}\rangle + J_0(J_0 - 1)|j, m_{max}\rangle = m_{min}(m_{min} - 1)|j, m_{max}\rangle$$

With $\lambda_j := j(j+1)$ we find $m_{max} = j = -m_{min}$ and

$$N_\pm = \sqrt{(j \mp m)(j \pm m + 1)}\, \mathrm{e}^{\mathrm{i}\phi_\pm}$$

4.5.2 The $so(4)$ symmetry of the H-atom

Classical Kepler problem:

$$E = \frac{\vec{p}^2}{2m} - \frac{\alpha}{r} , \qquad \alpha = GMm$$

with

$$\vec{\ell} = \vec{r} \times \vec{p} = mr^2\vec{\omega} = const. , \qquad \frac{\mathrm{d}}{\mathrm{d}t}\vec{e}_r = \vec{\omega} \times \vec{e}_r$$

and Newton equation

$$\vec{F} = \dot{\vec{p}} = -\frac{\alpha}{r^2}\,\vec{e}_r$$

Laplace-Runge-Lenz vector:

$$\vec{A} := \vec{p}\times\vec{\ell} - m\alpha\vec{e}_r = const.\,, \qquad \text{and} \qquad \vec{A}^{\,2} = m^2\alpha^2 + 2m\vec{\ell}^{\,2}E$$

Proofs:

- $\dot{\vec{A}} = \dot{\vec{p}}\times\vec{\ell} - m\alpha\dot{\vec{e}}_r = -\frac{\alpha}{r^2}\,\vec{e}_r\times(mr^2\vec{\omega}) - m\alpha\vec{\omega}\times\vec{e}_r = \vec{0}$
- $\vec{A}^{\,2} = (\vec{p}\times\vec{\ell})^2 - 2m\alpha\vec{e}_r\cdot(\vec{p}\times\vec{\ell}) + m^2\alpha^2 = \vec{p}^2\vec{\ell}^2 - (\vec{p}\cdot\vec{\ell})^2 - \frac{2m\alpha}{r}\vec{\ell}\cdot(\vec{r}\times\vec{p})) + m^2\alpha^2 =$
  $= \vec{p}^2\vec{\ell}^2 - \frac{2m\alpha}{r}\vec{\ell}^2 + m^2\alpha^2 = 2mE\vec{\ell}^2 + m^2\alpha^2$

Quantum mechanical hydrogen atom:
Hamiltonian:

$$H = \frac{1}{2m}\,\vec{P}^{\,2} - \alpha\,\frac{\vec{Q}}{|\vec{Q}|} \qquad \text{on} \qquad \mathcal{H} = L^2(\mathbb{R}^3)$$

Angular Momentum:

$$\vec{L} = \vec{Q}\times\vec{P}$$

Laplace-Runge-Lenz vector: (re-scaled and symmetrized)

$$\vec{A} := \frac{1}{2m\alpha}\left(\vec{P}\times\vec{L} - \vec{L}\times\vec{P}\right) - \frac{\vec{Q}}{|\vec{Q}|}$$

⇒ Two conserved vector operators

$$[H,\vec{L}] = \vec{0}\,, \qquad [H,\vec{A}] = \vec{0}$$

and

$$\vec{A}^2 = 1 + \frac{2}{m\alpha^2}\left(\hbar^2 + \vec{L}^{\,2}\right)H\,, \qquad \vec{L}\cdot\vec{A} = 0 = \vec{A}\cdot\vec{L}$$

Algebra:

$$\begin{aligned}
[A_i, A_j] &= \mathrm{i}\hbar\varepsilon_{ijk}L_k\left(-\tfrac{2H}{m\alpha^2}\right)\\
[L_i, L_j] &= \mathrm{i}\hbar\varepsilon_{ijk}L_k\\
[A_i, L_j] &= \mathrm{i}\hbar\varepsilon_{ijk}A_k = [L_i, A_j]
\end{aligned}$$

Consider subspace with fixed energy $E < 0$:

$$\mathcal{H}_E \subset L^2(\mathbb{R}^3) \qquad \text{with} \qquad \mathcal{H}_E := \{|\psi\rangle \in L^2(\mathbb{R}^3) | H|\psi\rangle = E|\psi\rangle\}$$

Redefine: $\vec{N} := \left(-\frac{2E}{m\alpha^2}\right)^{-1/2}\vec{A}, \qquad \vec{M} := \vec{L}$ then

$$\left.\begin{aligned}
[M_i, M_j] &= \mathrm{i}\hbar\varepsilon_{ijk}M_k\\
[M_i, N_j] &= \mathrm{i}\hbar\varepsilon_{ijk}N_k\\
[N_i, N_j] &= \mathrm{i}\hbar\varepsilon_{ijk}M_k
\end{aligned}\right\} \qquad so(4)\text{-algebra} \qquad \text{(see homework)}$$

Decoupling: $\vec{J} := \frac{1}{2}(\vec{M}+\vec{N})$ and $\vec{K} := \frac{1}{2}(\vec{M}-\vec{N})$

$$\left.\begin{aligned}
[J_i, J_j] &= \mathrm{i}\hbar\varepsilon_{ijk}J_k\\
[K_i, K_j] &= \mathrm{i}\hbar\varepsilon_{ijk}K_k\\
[K_i, J_j] &= 0
\end{aligned}\right\} \qquad so(4) \simeq so(3)\oplus so(3)$$

Consider UIR:

$$\begin{aligned}
\vec{K}^{\,2}|k, m_k\rangle &= k(k+1)\hbar^2|k, m_k\rangle\\
\vec{J}^2|j, m_j\rangle &= j(j+1)\hbar^2|j, m_j\rangle
\end{aligned}$$

Recall: $\vec{L}\cdot\vec{A} = 0 = \vec{M}\cdot\vec{K} = \vec{J}^{\,2} - \vec{K}^{\,2} \qquad \Rightarrow \qquad k = j$
Hence: $\mathcal{H}_E = D^k \otimes D^k$ product space of two UIR of $so(3)$
Product basis: $|k, m_k, m_j\rangle := |k, m_k\rangle \otimes |j, m_j\rangle$
Consider:

$$(\vec{J}^{\,2} + \vec{K}^{\,2})|k, m_k, m_j\rangle = 2k(k+1)\hbar^2|k, m_k, m_j\rangle$$

On the other hand:

$$\begin{aligned}\vec{J}^{\,2} + \vec{K}^{\,2} &= \tfrac{1}{2}(\vec{M}^{\,2} + \vec{N}^{\,2})\\ &= \tfrac{1}{2}\left(\vec{L}^{\,2} - \tfrac{m\alpha^2}{2E}\vec{A}^{\,2}\right)\\ &= \tfrac{1}{2}\left(\vec{L}^{\,2} - \tfrac{m\alpha^2}{2E}\left(1 + \tfrac{2E}{m\alpha^2}(\hbar^2 + \vec{L}^2)\right)\right)\\ &= -\tfrac{m\alpha^2}{4E} - \tfrac{\hbar^2}{2}\end{aligned}$$

$\Rightarrow$

$$-\frac{m\alpha^2}{4E} = \frac{\hbar^2}{2} + 2k(k+1)\hbar^2 = \frac{\hbar^2}{2}(4k^2 + 4k + 1) = \frac{\hbar^2}{2}(2k+1)^2$$

$\Rightarrow$

$$E = -\frac{m\alpha^2}{2\hbar^2}\frac{1}{(2k+1)^2} = -\frac{m\alpha^2}{2\hbar^2}\frac{1}{n^2} \qquad \text{with} \qquad n := 2k+1$$

Note: $k \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\} \qquad \Rightarrow \qquad n \in \{1, 2, 3, 4, \ldots\}$

Degeneracy: $\dim\mathcal{H}_E = \dim(D^k \otimes D^k) = (2k+1)^2 \qquad \Rightarrow \qquad E = E_n$ is n^2 degenerate.

Angular momentum: $\vec{L} = \vec{J} + \vec{K}$ coupling of j and k
$\Rightarrow$

$$\begin{aligned}\ell &\in \{|j-k|, |j-k|+1, \ldots j+k\} \qquad \text{but} \qquad k = j\\ &= \{0, 1, 2, \ldots, 2k\} \qquad \text{only integer} \qquad \ell\end{aligned}$$

With $n = \ell + 1 + n_r$ and $m = m_k + m_j$ we change to new

$$|k, m_k, m_j\rangle \qquad \longrightarrow \qquad |n, \ell, m\rangle$$

Eigenstates with

$$\begin{aligned}H|n,\ell,m\rangle &= E_n|n,\ell,m\rangle\\ \vec{L}^2|n,\ell,m\rangle &= \hbar^2\ell(\ell+1)|n,\ell,m\rangle\\ L_z|n,\ell,m\rangle &= m\hbar|n,\ell,m\rangle\end{aligned}$$

Comments:

- For $E > 0$ one obtains a $SO(3,1) = SO(3) \otimes SO(2,1)$ symmetry. See later for an algebraic approach via $so(2,1) \simeq su(1,1)$ (spectrum-generating algebra).
  In essence: $\vec{N} := \left(\frac{2E}{m\alpha^2}\right)^{-1/2}\vec{A}$

  $$\left.\begin{aligned}[M_i, M_j] &= \mathrm{i}\hbar\varepsilon_{ijk}M_k\\ [M_i, N_j] &= \mathrm{i}\hbar\varepsilon_{ijk}N_k\\ [N_i, N_j] &= -\mathrm{i}\hbar\varepsilon_{ijk}M_k\end{aligned}\right\} \qquad so(3,1)\text{-algebra}$$

  Bertrand's Theorem: There are only two types of central-force (radial) scalar potentials with the property that all bound orbits are also closed orbits.

- The 3-D Kepler problem may also be mapped onto a 4-D harmonic oscillator problem via the so-called Kustaanheimo-Stiefel transformation (c.f. Homework Problem 8). This Newton-Hook duality was already know to both in 17th century.

- A fixed $SO(4)$-UIR spans the subspace $\mathcal{H}_E$ corresponding to a single energy shell of a bound state (fixed $n = 2k+1$ and varying ℓ, m).

- An irreducible representation of $SO(4,1)$ spans the full bound spectrum. The group $SO(4,1)$ is also called de Sitter group (Willem de Sitter 1872–1934). de Sitter space is a maximally symmetric Lorentzian manifold with constant positive scalar curvature. Embed S^3 in $\mathbb{R}^4$
- An $SO(3,2)$ irreducible representation spans the full continuous spectrum. The group $SO(3,2)$ is also called anti-de Sitter group. Anti-de Sitter space is a maximally symmetric Lorentzian manifold with constant negative scalar curvature.
- $SO(4,2)$ is called the full dynamical group of the Kepler (or Hydrogen atom problem). It is the smallest group whose irreducible representations span both the continuous and the discrete spectrum.

Some proofs:

$\vec{L}\cdot\vec{A} = 0 = \vec{A}\cdot\vec{L}$ is obvious as $\vec{L}\cdot(\vec{P}\times\vec{L}) = 0$ and $\vec{L}\cdot\vec{Q} = 0$

With $\vec{L}\times\vec{P} = 2\mathrm{i}\hbar\vec{P} - \vec{P}\times\vec{L}$ and $R := |\vec{Q}|$ follows

$$\vec{A} = \frac{1}{\alpha m}\left(\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P}\right) - \frac{\vec{Q}}{R}$$

$\Rightarrow$

$$\alpha^2 m^2(\vec{A}^2 - 1) = (\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P})^2 - \alpha m(\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P})\cdot\frac{\vec{Q}}{R} - \alpha m\frac{\vec{Q}}{R}\cdot(\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P})$$

Using following relations (proofs are below)

$$\begin{array}{lcl}
(\vec{P}\times\vec{L})\cdot(\vec{P}\times\vec{L}) = \vec{P}^2\vec{L}^2 & & \\
(\vec{P}\times\vec{L})\cdot\vec{P} = 2\mathrm{i}\hbar\vec{P}^2 & & (\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P})^2 = \vec{P}^2(\vec{L}^2 + \hbar^2) \\
\vec{P}\cdot(\vec{P}\times\vec{L}) = 0 & \Rightarrow & (\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P})\cdot\vec{Q} = \vec{L}^2 + \mathrm{i}\hbar\vec{P}\cdot\vec{Q} \\
(\vec{P}\times\vec{L})\cdot\vec{Q} = \vec{L}^2 + 2\mathrm{i}\hbar\vec{P}\cdot\vec{Q} & & \vec{Q}\cdot(\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P}) = \vec{L}^2 - \mathrm{i}\hbar\vec{Q}\cdot\vec{P} \\
\vec{Q}\cdot(\vec{P}\times\vec{L}) = \vec{L}^2 & &
\end{array}$$

follows

$$\alpha^2 m^2(\vec{A}^2 - 1) = \vec{P}^2(\vec{L}^2 + \hbar^2) - 2\alpha m\frac{\vec{L}^2}{R} - \mathrm{i}\hbar\alpha m\underbrace{\left(\vec{P}\cdot\frac{\vec{Q}}{R} - \frac{\vec{Q}}{R}\cdot\vec{P}\right)}_{-2\mathrm{i}\hbar/R} = 2mH\left(\vec{L}^2 + \hbar^2\right)$$

Auxiliary formulas:

- $(\vec{P}\times\vec{L})\cdot(\vec{P}\times\vec{L}) = \varepsilon_{ijk}P_jL_k\varepsilon_{ilm}P_lL_m = \varepsilon_{ijk}\varepsilon_{ilm}P_jL_kP_lL_m =$
  $= \varepsilon_{ijk}\varepsilon_{ilm}P_j(P_lL_k + \mathrm{i}\hbar\varepsilon_{klr}P_r)L_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(P_jP_lL_kL_m + \mathrm{i}\hbar\varepsilon_{klr}P_jP_rL_m) =$
  $= \vec{P}^2\vec{L}^2 + \mathrm{i}\hbar\varepsilon_{kjr}P_jP_rL_k - P_j\vec{P}\cdot\vec{L}L_j - \mathrm{i}\hbar\varepsilon_{kkr}P_jP_rL_j) = \vec{P}^2\vec{L}^2$
- $(\vec{P}\times\vec{L})\cdot\vec{P} = \varepsilon_{ijk}P_iL_jP_k = \varepsilon_{ijk}P_i(P_kL_j + \mathrm{i}\hbar\varepsilon_{jkl}P_l) = \mathrm{i}\hbar 2\delta_{il}P_iP_l = 2\mathrm{i}\hbar\vec{P}^2$
- $\vec{P}\cdot(\vec{P}\times\vec{L}) = \varepsilon_{ijk}P_iP_jL_k = 0$
- $(\vec{P}\times\vec{L})\cdot\vec{Q} = \varepsilon_{ijk}P_iL_jQ_k = \varepsilon_{ijk}P_i(Q_kL_j + \mathrm{i}\hbar\varepsilon_{jkl}Q_l) = \varepsilon_{ijk}Q_kP_iL_j + \mathrm{i}\hbar 2\delta_{il}P_iQ_l =$
  $= \vec{L}^2 + 2\mathrm{i}\hbar\vec{P}\cdot\vec{Q}$
- $\vec{Q}\cdot(\vec{P}\times\vec{L}) = \varepsilon_{ijk}Q_iP_jL_k = \vec{L}^2$

- $(\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P})^2 = (\vec{P}\times\vec{L})\cdot(\vec{P}\times\vec{L}) - \mathrm{i}\hbar(\vec{P}\times\vec{L})\cdot\vec{P} - \mathrm{i}\hbar\vec{P}\cdot(\vec{P}\times\vec{L}) - \hbar^2\vec{P}^2 =$
  $= \vec{P}^2\vec{L}^2 - \mathrm{i}2\mathrm{i}\hbar^2\vec{P}^2 - \hbar^2\vec{P}^2 = \vec{P}^2(\vec{L}^2 + \hbar^2)$
- $(\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P})\cdot\vec{Q} = \vec{L}^2 + 2\mathrm{i}\hbar\vec{P}\cdot\vec{Q} - \mathrm{i}\hbar\vec{P}\cdot\vec{Q} = \vec{L}^2 + \mathrm{i}\hbar\vec{P}\cdot\vec{Q}$
- $\vec{Q}\cdot(\vec{P}\times\vec{L} - \mathrm{i}\hbar\vec{P}) = \vec{L}^2 - \mathrm{i}\hbar\vec{Q}\cdot\vec{P}$
- $\vec{P}\cdot\frac{\vec{Q}}{R} - \frac{\vec{Q}}{R}\cdot\vec{P} = \frac{\hbar}{\mathrm{i}}(\vec{\nabla}\cdot\vec{Q})\frac{1}{R} + \frac{\hbar}{\mathrm{i}}\vec{Q}\cdot(\vec{\nabla}\frac{1}{R}) = \frac{\hbar}{\mathrm{i}}\frac{3}{R} - \frac{\hbar}{\mathrm{i}}\vec{Q}\cdot\frac{\vec{Q}}{R^3} = \frac{\hbar}{\mathrm{i}}\frac{2}{R}$

Finally the proof for $[\vec{A}, H] = 0$

With

$$\vec{P} \times \vec{L} = P^2\vec{Q} - (\vec{P}\cdot\vec{Q})\vec{P} + \mathrm{i}\hbar\vec{P} \qquad \text{and} \qquad \vec{L} \times \vec{P} = -P^2\vec{Q} + (\vec{P}\cdot\vec{Q})\vec{P} + \mathrm{i}\hbar\vec{P}$$

follows

$$\vec{A} = \frac{1}{m\alpha}\left(P^2\vec{Q} - (\vec{P}\cdot\vec{Q})\vec{P}\right) - \frac{\vec{Q}}{R}$$

Using following formulas

- $[\vec{P}, 1/R] = \mathrm{i}\hbar\frac{\vec{Q}}{R^3}$
- $[P^2, 1/R] = \mathrm{i}\hbar\frac{1}{R^3}(\vec{Q}\cdot\vec{P}) + \mathrm{i}\hbar(\vec{P}\cdot\vec{Q})\frac{1}{R^3}$
- $[(\vec{P}\cdot\vec{Q})\vec{P}, 1/R] = \mathrm{i}\hbar(\vec{P}\cdot\vec{Q})\frac{1}{R^3} + \mathrm{i}\hbar\frac{1}{R}\vec{P}$

one finds

$$\begin{aligned}
m\alpha[\vec{A}, \tfrac{1}{R}] &= [P^2\vec{Q}, \tfrac{1}{R}] - [(\vec{P}\cdot\vec{Q})\vec{P}, \tfrac{1}{R}] \\
&= \mathrm{i}\hbar\tfrac{1}{R^3}(\vec{Q}\cdot\vec{P})\vec{Q} + \mathrm{i}\hbar(\vec{P}\cdot\vec{Q})\tfrac{\vec{Q}}{R^3} - \mathrm{i}\hbar(\vec{P}\cdot\vec{Q})\tfrac{\vec{Q}}{R^3} - \mathrm{i}\hbar\tfrac{1}{R}\vec{P} \\
&= \mathrm{i}\hbar\tfrac{1}{R^3}(\vec{Q}\cdot\vec{P})\vec{Q} - \mathrm{i}\hbar\tfrac{1}{R}\vec{P}
\end{aligned}$$

Consider now

$$\begin{aligned}
[\vec{A}, P^2] &= \tfrac{1}{\alpha m}\left([P^2\vec{Q}, P^2] - [(\vec{P}\cdot\vec{Q})\vec{P}, P^2]\right) - [\tfrac{\vec{Q}}{R}, P^2] \\
&= \tfrac{1}{\alpha m}\left(P^2[\vec{Q}, P^2] - [(\vec{P}\cdot\vec{Q}), P^2]\vec{P}\right) - \tfrac{1}{R}[\vec{Q}, P^2] - [\tfrac{1}{R}, P^2]\vec{Q} \\
&= \tfrac{1}{\alpha m}\underbrace{\left(P^2 2\mathrm{i}\hbar\vec{P} - \vec{P}\cdot[\vec{Q}, P^2]\vec{P}\right)}_{=0} - \tfrac{1}{R}[\vec{Q}, P^2] - [\tfrac{1}{R}, P^2]\vec{Q} \\
&= -\tfrac{1}{R}2\mathrm{i}\hbar\vec{P} + \mathrm{i}\hbar\tfrac{1}{R^3}(\vec{Q}\cdot\vec{P})\vec{Q} + \mathrm{i}\hbar(\vec{P}\cdot\vec{Q})\tfrac{\vec{Q}}{R^3} \qquad \text{with} \qquad [\vec{P}\cdot\vec{Q}, \tfrac{1}{R^3}] = 3\mathrm{i}\hbar\tfrac{1}{R^3} \\
&= -2\mathrm{i}\hbar\tfrac{1}{R}\vec{P} + \mathrm{i}\hbar\tfrac{1}{R^3}\left((\vec{Q}\cdot\vec{P}) + \underbrace{(\vec{P}\cdot\vec{Q}) + 3\mathrm{i}\hbar}_{(\vec{Q}\cdot\vec{P})}\right)\vec{Q} \\
&= -2\mathrm{i}\hbar\tfrac{1}{R}\vec{P} + 2\mathrm{i}\hbar\tfrac{1}{R^3}(\vec{Q}\cdot\vec{P})\vec{Q} \\
&= 2m\alpha[\vec{A}, \tfrac{1}{R}]
\end{aligned}$$

Hence

$$\frac{1}{2m}[\vec{A}, P^2] = [\vec{A}, \alpha/R] \qquad \Rightarrow \qquad [\vec{A}, H] = 0$$

Good References:
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*** End of Lecture 4 ***