2.3 Representations of an Abstract Group

Let V be a d-dimensional linear vector space (real or complex).
That is for $\vec{u}, \vec{v} \in V \Rightarrow \vec{u}+\vec{v} \in V$ and $\alpha\vec{v} \in V$, $\alpha \in \mathbb{R}$ or $\mathbb{C}$ with

$$\begin{array}{ll} \vec{v}+\vec{u}=\vec{u}+\vec{v} & \alpha(\beta\vec{v})=(\alpha\beta)\vec{v} \\ (\vec{v}+\vec{u})+\vec{w}=\vec{v}+(\vec{u}+\vec{w}) & 1\vec{v}=\vec{v},\ 0\vec{v}=\vec{0},\ -1\vec{v}=-\vec{v} \\ \vec{v}+\vec{0}=\vec{v} & (\alpha+\beta)\vec{v}=\alpha\vec{v}+\beta\vec{v} \\ \vec{v}+(-\vec{v})=\vec{0} & \alpha(\vec{v}+\vec{u})=\alpha\vec{v}+\alpha\vec{u} \end{array}$$

Let $\mathcal{D}: V \to V$ be a linear invertible transformation (operator) acting on V

$$\mathcal{D}(\alpha\vec{u}+\beta\vec{v}) = \alpha\mathcal{D}\vec{u}+\beta\mathcal{D}\vec{v}$$

Definition: A *d-dimensional linear representation* of a group G is a group homomorphism

$$\mathcal{D}: \begin{array}{l} G \to GL(V) := \text{ group of linear invertible transformations acting on } V \\ g \mapsto \mathcal{D}(g) \end{array}$$

with group law

$$\mathcal{D}(g_1 g_2) = \mathcal{D}(g_1)\mathcal{D}(g_2)$$

Remarks:

- Usually for finite-dimensional reps $GL(V) = GL(d,\mathbb{C})$ set of linear complex-valued $d \times d$ matrices
- $d = \infty$ is allowed, for example $V = L^2(\mathbb{R}^3)$ Hilbert space $\Rightarrow GL(V)$ is set of linear operators acting on V
- V is called *representation space*
- Let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d\}$ be *complete orthonormal basis* in V with scalar product

  $$(\vec{e}_i, \vec{e}_j) = \delta_{ij}$$

  Then

  $$D_{ij}(g) := (\vec{e}_i, \mathcal{D}(g)\vec{e}_j)$$

  are the matrix elements of the *matrix representation* $D(g)$.
  Often no difference is made between $\mathcal{D}(g)$ and $D(g)$
- Is $D(g)$ linear operator $\forall\, g \in G \Leftrightarrow$: *linear representation*
  Non-linear representations are also called *realisations*
- Exists a similarity transformation S such that

  $$\tilde{D}(g) := S^{-1}D(g)S \qquad \forall\, g \in G$$

  is also a representation of G, then $\tilde{D}$ and D are called *equivalent* representations (change of basis).
- Notation: $D^d(g)$ usually stands for a d-dimensional representation, $\{D_i(g)\}$ or $\{D^i(g)\}$ stands for set of reps. enumerated by an index i. Known example for rotation group is $\ell = 0, 1, 2, 3, \ldots$ with dimension $d_\ell = 2\ell + 1$.

Unitary representation:

$$D(g) \text{ unitary } \forall\, g \in G \qquad :\Leftrightarrow \qquad (D(g)\vec{u}, D(g)\vec{v}) = (\vec{u}, \vec{v})\ \forall\, g \in G \text{ and } \forall\, \vec{u}, \vec{v} \in V$$

$$\Rightarrow D(g^{-1}) = D^\dagger(g) = D^{-1}(g)$$

Faithful representation: Homomorphism is an isomorphism

$$g_1 \neq g_2 \qquad \Rightarrow D(g_1) \neq D(g_2)$$

Trivial representation: unitary but not faithful

$$D^{\text{trivial}}(g) := 1 \; \forall \, g \in G$$

Regular representation: $G = \{g_1, g_2, \ldots, g_n\}$ finite

$$g g_j =: \sum_{i=1}^{n} D^{\text{reg}}_{ij}(g) g_i$$

$D^{\text{reg}}(g)$ is an $n \times n$ matrix with a single 1 and rest zeros in each row and column
n-dimensional faithful representation (group table)

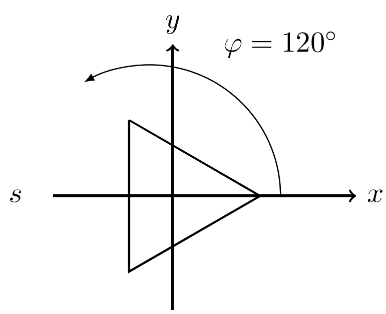
2.4 Representations of D_3

Recall: $D_3 = \{e, d, d^2, s, sd, sd^2\}$, $d^3 = e = s^2$, $sd = d^{-1}s$

- $D^1_s(g) := 1$ 1-dimensional symmetric reps. = trivial reps.
- $D^1_a(g)$ 1-dimensional anti-symmetric reps. with

$$D^1_a(g) := \begin{cases} 1 & g \in \{e, d, d^2\} = E \\ -1 & g \in \{s, sd, sd^2\} = D \end{cases}$$

- $D^2(g)$ 2-dimensional reps. explicitly constructed for generators d, s on $\mathbb{R}^2$



Obviously:
$D^2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$D^2(d) = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
$D^2(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Proof:
$D^2(d^2) = D^2(d)D^2(d) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \left[D^2(d)\right]^\dagger = D^2(d^{-1})$ unitary
$D^2(d^3) = D^2(d)D^2(d)D^2(d) = D^2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$D^2(sd) = D^2(s)D^2(d) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = D^2(d^{-1})D^2(s) = D^2(d^{-1}s)$

$D^2(sd^2) = D^2(sd)D^2(d) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = D^2(d^{-1})D^2(d^{-1}s) = D^2(d^{-1})D^2(s) = D^2(d^{-2}s)$

$\Rightarrow$ 2-dim. faithful and unitary reps.

Comment: 3 conjugacy classes $\Rightarrow$ 3 unitary irreducible reps. (see later)

2.5 Properties of Representations for Finite Groups

Maschke's Theorem:
Each representation of a finite group is equivalent to a unitary representation.

Proof: See Tutorial

Comment: Can be extended to continuous (uni-modular) groups with invariant Haar measure. In physics we usually deal with unitary irreducible representations **UIR**.

Reducible Representation:
Let $D(g)$ be a d-dimensional reps. in V, $\dim V = d$.
If there exists an invariant subspace $U \subset V$ with $\dim U < \dim V$, that is, with $\vec{u} \in U \Rightarrow D(g)\vec{u} \in U$ for all $g \in G$, then the representation is called *reducible.*

The representation matrices are of the form

$$D(g) = \left(\begin{array}{c|c} D_1(g) & R(g) \\ \hline 0 & D_2(g) \end{array} \right)$$

Irreducible Representation:
If there exists NO invariant subspace in V the representation is called *irreducible.*

Theorem:
Let $D(g)$ be unitary and reducible with invariant subspace U. Then $U^\perp$ is also invariant subspace and $V = U \oplus U^\perp$. That is $R(g) = 0$ for unitary reducible reps.

Proof: Let $\vec{u} \in U$ and $\vec{w} \in U^\perp$ then for all $g \in G$ $D(g)\vec{u} \in U$
$\Rightarrow$ $0 = (D(g)\vec{u}, \vec{w}) = (\vec{u}, D^\dagger(g)\vec{w}) = (\vec{u}, D(g^{-1})\vec{w})$ for all $g \in G$
$\Rightarrow$ $(\vec{u}, D(g)\vec{w}) = 0$ for all $g \in G$

Conclusion: Representation matrices of unitary reducible reps. are (in a proper basis) block-diagonal

$$D(g) = \left(\begin{array}{c|c} D_1(g) & 0 \\ \hline 0 & D_2(g) \end{array} \right) \qquad \text{or} \qquad D = D_1 \oplus D_2$$

Fully reducible representations:
Can the representation space of a reducible representation D be decomposed into invariant irreducible subspaces then D is called *fully reducible*

$$D = r_1\, D_1 \oplus r_2\, D_2 \oplus \cdots \oplus r_s D_s$$

Here $r_i \in \mathbb{N}$ denotes the multiplicity of occurrence of irr. reps. D_i in D
The representation matrices are block-diagonal

$$D(g) = \left(\begin{array}{c|c|c|c} D_1(g) & 0 & \cdots & \\ \hline 0 & D_1(g) & 0 & \ddots \\ \hline \vdots & 0 & D_2(g) & 0 \\ \hline & \vdots & 0 & \ddots \end{array} \right)$$

Comments:
Unitary reps. are either irreducible or fully reducible.
All reps. of finite groups are either irreducible or fully reducible.

Example: The *natural* reps. of $S_3 = \{e, a, b, c, d, f\}$
Recall:

$$e := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad a := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$c := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad d := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad f := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3,\}$ be the natural basis of $\mathbb{R}^3$ and $P := \begin{pmatrix} 1 & 2 & 3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$ then

$$D^{\text{nat}}(P) := \sum_{i=1}^{3} \vec{e}_{\pi_i} \vec{e}_i^{\,\mathrm{T}} \qquad \text{permutation of base vectors} \qquad \vec{e}_i \to \vec{e}_{\pi_i}$$

Explicit

$$D^{\text{nat}}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D^{\text{nat}}(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D^{\text{nat}}(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$D^{\text{nat}}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D^{\text{nat}}(d) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad D^{\text{nat}}(f) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

1-dim. invariant subspace: $\vec{v} := \frac{1}{\sqrt{3}}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3)$ obvious
2-dim. subspace orthogonal to $\vec{v}$:

$$\vec{u}_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \vec{u}_2 := \vec{v} \times \vec{u}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \qquad \text{obvious}$$

Change of basis:

$$S := (\vec{u}_1, \vec{u}_2, \vec{v}) = \begin{pmatrix} 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad \text{obviously } S^\dagger S = 1$$

Equivalent reps.:

$$\tilde{D}^{\text{nat}}(a) := S^\dagger D^{\text{nat}}(a) S = \left(\begin{array}{cc|c} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = D^2(d) \oplus D^1_s(d)$$

$$\tilde{D}^{\text{nat}}(c) := S^\dagger D^{\text{nat}}(c) S = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = D^2(s) \oplus D^1_s(s)$$

Remember $S_3 \simeq D_3$ with $a \simeq d$ and $c \simeq s$
Hence $D^{\text{nat}} = D^2 \oplus D^1_s$

2.6 First and Second Schur Lemma

2.6.1 First Lemma

1. Schur Lemma:
Let D be an *irreducible* matrix representation of a group G in representation space V and M be a matrix representing an operator in V such that

$$MD(g) = D(g)M \qquad \forall\, g \in G$$

then

$$M = \lambda\,\mathbf{1}\,, \qquad \lambda = const.$$

That is, if M has G-symmetry then it is proportional to the unit matrix $\mathbf{1}$ in V.

Proof:
Let $\vec{x} \in V$ be eigenvector of M, $M\vec{x} = \lambda\vec{x}$
$\Rightarrow \vec{x}_g := D(g)\vec{x}$ is also eigenvector with same eigenvalue for all $g \in G$ as $[M, D(g)] = 0$
$\Rightarrow \exists\, U_\lambda \subseteq V$ such that $D(g)U_\lambda = U_\lambda$ is invariant subspace for eigenvalue λ
But D is irreducible and therefore $U_\lambda = V \Rightarrow M = \lambda\mathbf{1}$ in V

Comments:

- If the only matrix commuting with all $D(g)$ is proportional to the unit matrix then D is irreducible reps.

- All irreducible reps. of abelian groups are 1-dimensional
  D irreducible $\Rightarrow D(g_i)D(g) = D(g)D(g_i)$ for all $g, g_i \in G$. So let $M = D(g_i) \Rightarrow D(g_i) = \lambda\mathbf{1}$ and irreducible $\Rightarrow$ 1-dimensional

- Unitary irreducible representations (UIR) of abelian groups are of the form

$$D(g) = \mathrm{e}^{\mathrm{i}\alpha(g)}\,, \qquad \alpha : \begin{array}{l} G \to [0, 2\pi[\\ g \mapsto \alpha(g) \end{array} \qquad \text{with} \qquad \alpha(g_1 g_2) = \alpha(g_1) + \alpha(g_2) \bmod 2\pi$$

2.6.2 Second Lemma

2. Schur Lemma:
Let D^1 and D^2 be non-equivalent UIR of dimension d_1 and d_2. Then any rectangular $d_1 \times d_2$ matrix M which obeys

$$MD^1(g) = D^2(g)M \qquad \forall\, g \in G$$

is the null matrix

$$M = 0$$

Proof:
Consider adjoint equation ${D^1}^\dagger(g)M^\dagger = M^\dagger {D^2}^\dagger(g)$ then
$D^1(g^{-1})M^\dagger = M^\dagger D^2(g^{-1}) \Rightarrow D^1(g)M^\dagger = M^\dagger D^2(g)$ for all $g \in G$
$\Rightarrow MD^1(g)M^\dagger = D^2(g)MM^\dagger = MM^\dagger D^2(g)$
1. Lemma $\Rightarrow MM^\dagger = \lambda\,\mathbf{1}$

Case $d_1 = d_2$: Let $\det M \neq 0$, then there exist a M^{-1} such that $D^1(g) = M^{-1}D^2(g)M$
$\Rightarrow D^1$ and D^2 are equivalent, which contradicts assumption $\Rightarrow \det M = 0$
$\Rightarrow \det MM^\dagger = |\lambda|^{d_2} = 0 \Rightarrow \lambda = 0 \Rightarrow MM^\dagger = 0 \Rightarrow M = 0$

Case $d_1 < d_2$ (without loss of generality): Complete M to $d_2 \times d_2$ matrix $\tilde{M} := (M|0)$ with additional zero columns $\Rightarrow \tilde{M}\tilde{M}^\dagger = (M|0)\begin{pmatrix} M^\dagger \\ 0 \end{pmatrix} = MM^\dagger = \lambda\mathbf{1} \Rightarrow \tilde{M} = 0 \Rightarrow M = 0.$

Both Schur lemmata in a nut shell:
For UIR

$$\boxed{\begin{array}{ll} M D^i(g) = D^j(g) M \qquad \Rightarrow \qquad M = \lambda\, \delta_{ij} \\ \text{with} \qquad \delta_{ij} = \left\{ \begin{array}{ll} 0 & \text{for} \quad i \neq j \text{ inequivalent reps.} \\ 1 & \phantom{\text{for}} \quad i = j \text{ equivalent reps.} \end{array} \right. \end{array}}$$

2.6.3 Application to Eigenvalue Problems

Let H be linear operator in V. For example: $H = \vec{P}^2/2m + V(|\vec{Q}|)$, $V = L^2(\mathbb{R}^3)$.
Let D be unitary reducible reps of G in V. For example $G = SO(3)$, $D(g(\vec{\omega})) = \exp\{-\mathrm{i}\vec{\omega}\cdot\vec{L}\}$.

We say H has G-symmetry if

$$[D(g), H] = 0 \qquad \forall\, g \in G \qquad \Leftrightarrow \qquad D(g)H = HD(g)$$

Problem: Find eigenvalues and eigenvectors of H in V, $|H - \lambda\mathbf{1}| = 0$.

Note: D is completely reducible

$$D(g) = c_1\, D^1(g) + c_2\, D^2(g) + \ldots + c_n\, D^n(g)\,, \qquad n \leq \infty$$

$$D^i(g) : \text{UIR of dimension } d_i$$

$$c_i : \text{multiplicity of } D^i \text{ in } D$$

Reduction of Problem:
With suitable basis in V the reps. matrix for D is block diagonal
Example: $D(g) = 2\, D^1(g) + D^2(g)$

$$D(g) = \left(\begin{array}{cc|c} D^1(g) & 0 & 0 \\ 0 & D^1(g) & 0 \\ \hline 0 & 0 & D^2(g) \end{array} \right) \begin{array}{l} \Sigma^1 \\ \\ \Sigma^2 \end{array}$$

Conglomerate: Σ^i inv. subspace of V containing all the UIR D^i.
Write H in that basis

$$H = \left(\begin{array}{cc|c} H^{(1)}_{11} & H^{(1)}_{12} & H^{(12)}_{11} \\ H^{(1)}_{21} & H^{(1)}_{22} & H^{(12)}_{21} \\ \hline H^{(21)}_{11} & H^{(21)}_{22} & H^{(2)}_{11} \end{array} \right) \begin{array}{l} \Sigma^1 \\ \\ \Sigma^2 \end{array}$$

In general:
$H^{(i)}_{lm} = d_i \times d_i$ matrix, lm element of a submatrix of H in Σ^i
$l = m$ submatrix of H in subspace belonging to a fixed D^i
$l \neq m$ overlap of m-th and l-th UIR D^i, l and $m \in \{1, 2, \ldots, c_i\}$
$H^{(ij)}_{lm} = d_i \times d_j$ matrix, overlap of l-th UIR D^i with m-th UIR $D^i \neq D^j$

Symmetry of H: $D(g)H = HD(g)$ for all $g \in G$

$$\Rightarrow \left\{ \begin{array}{lll} D^i(g) H^{(i)}_{lm} = H^{(i)}_{lm} D^i(g) & \stackrel{1.\text{SL}}{\Rightarrow} & H^{(i)}_{lm} = h^{(i)}_{lm} \mathbf{1}_{d_i} \\ D^i(g) H^{(ij)}_{lm} = H^{(ij)}_{lm} D^j(g) & \stackrel{2.\text{SL}}{\Rightarrow} & H^{(ij)}_{lm} = 0 \end{array} \right.$$

In our example:

$$H = \left(\begin{array}{cc|c} h^{(1)}_{11} \mathbf{1}_{d_1} & h^{(1)}_{12} \mathbf{1}_{d_1} & 0 \\ h^{(1)}_{21} \mathbf{1}_{d_1} & h^{(1)}_{22} \mathbf{1}_{d_1} & 0 \\ \hline 0 & 0 & h^{(2)}_{11} \mathbf{1}_{d_2} \end{array} \right)$$

In subspace Σ^i operator H consists of $c_i \times c_i$ blocks of dimension d_i being diagonal matrices. In general with definition $\left(\tilde{H}^{(i)}\right)_{lm} := h^{(i)}_{lm} \Rightarrow \tilde{H}^{(i)}$ is $c_i \times c_i$ matrix

$$\Rightarrow \qquad H = \sum_{i=1}^{n} \left(\tilde{H}^{(i)} \times \mathbf{1}_{d_i}\right) \qquad V = \bigoplus_{i=1}^{n} V_i\,, \qquad \dim V_i = c_i d_i$$

In our example:

$$H = \left(\begin{array}{c|c} \tilde{H}^{(1)} \times \mathbf{1}_{d_1} & 0 \\ \hline 0 & \tilde{H}^{(2)} \times \mathbf{1}_{d_2} \end{array}\right)$$

with

$$\tilde{H}^{(1)} = \begin{pmatrix} h^{(1)}_{11} & h^{(1)}_{12} \\ h^{(1)}_{21} & h^{(1)}_{22} \end{pmatrix}, \quad c_1 = 2 \qquad \text{and} \qquad \tilde{H}^{(2)} = h^{(2)}_{11}\,, \quad c_2 = 1$$

Conclusion:
With a suitable basis (also within the subspaces Σ^i) the eigenvalue problem for H can be reduced to n eigenvalue problems of the form

$$\left|\tilde{H}^{(i)} - \lambda^{(i)}\mathbf{1}_{c_i}\right| = 0$$

$n =$ number of different UIR of symmetry of H occurring in V.

Comments:

- $D(g) = c\, D^{\text{trivial}}(g)$, $c = \dim V$ only trivial representation
  $\Rightarrow$ no symmetry $\Rightarrow$ no simplification
- $D(g) = D^i(g)$ is already UIR $\Rightarrow H = \lambda\mathbf{1}$, only one eigenvalue, problem solved
- D^i appears only once in decomposition, that is, $c_i = 1$, then invariant subspace Σ^i is also eigenspace of $H \Rightarrow$ degeneracy due to symmetry
  Same eigenvalue may accidentally occur also in other subspaces
  $\Rightarrow$ accidental degeneracy (usually a sign of an additional hidden symmetry)

Summary:

1. Choose suitable symmetry group and its representation in V
   Aim is to have c_i's as small as possible $\Rightarrow d_i$ as large as possible as $\dim V = \sum_{i=1}^{n} c_i d_i$
   higher symmetry groups have higher-dim. UIR ($C_n \subset D_n \subset S_n$)
   $\Rightarrow$ "higher" simplification
2. Decompose representation into UIRs
3. Choose symmetry adopted basis in subspaces Σ

Know example from QM

$H = \dfrac{\vec{P}^2}{2m} + V(|\vec{Q}|)$, $V = L^2(\mathbb{R}^3) = L^2(\mathbb{R}^+) \otimes L^2(S^2)$

$L^2(S^2) = \bigoplus_{l=0}^{\infty} D^l$, $d_l = 2l+1$, $c_l = 1$

$D(g) = \exp\{-\mathrm{i}\vec{\omega}\cdot\vec{L}\}$ with $[H, D(g)] = 0$ for all $g \in SO(3)$ as $[H, \vec{L}] = \vec{0}$.

$\Rightarrow H = \sum_{l=0}^{\infty} H^l_r \otimes \mathbf{1}_{2l+1}$ with

$$H^l_r = -\frac{\hbar^2}{2m}\partial_r^2 + \frac{\hbar^2 l(l+1)}{2m} + V(r)$$

in suitable basis

$$\Psi_l(\vec{r}) = \sum_{l=0}^{\infty} r R_l(r) \sum_{l=-m}^{m} Y_{lm}(\theta, \varphi)$$

Note $\langle \theta\varphi | lm \rangle = Y_{lm}(\theta, \varphi)$ and $D^l = \sum_{m=-l}^{m} |lm\rangle\langle lm|$

An explicit example will be worked out in the Tutorial in Exercise 4.

2.7 Orthogonality of Representations and Characters

2.7.1 Orthogonality of UIR

Theorem: Let $D^i(g)$ and $D^j(g)$ be matrices of two UIR of dimension d_i and d_j for a group G, $g \in G$. Then the following orthogonality relation of the matrix elements holds:

$$\boxed{\frac{1}{n} \sum_{g \in G} D^i_{\mu\nu}(g) D^{j\,*}_{\rho\sigma}(g) = \frac{1}{d_i} \delta_{ij} \delta_{\mu\rho} \delta_{\nu\sigma}}\,,$$

where $n = \operatorname{ord} G$ and

$$\delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j \text{ inequivalent reps.} \\ 1 & \phantom{\text{for}} \quad i = j \text{ equivalent reps.} \end{cases}$$

Proof: Consider

$$M := \sum_{g \in G} D^i(g)\, X\, D^j(g^{-1})$$

with X being an arbitrary $d_i \times d_j$ matrix. Then for all $g_0 \in G$

$$\begin{aligned} D^i(g_0) M &= \sum_{g \in G} D^i(g_0) D^i(g)\, X\, D^j(g^{-1}) D^j(g_0^{-1}) D^j(g_0) \\ &= \sum_{g \in G} D^i(g_0 g)\, X\, D^j((g g_0)^{-1}) D^j(g_0) \\ &= M D^j(g_0) \end{aligned}$$

and therefore $M = \lambda\, \mathbf{1} \delta_{ij}$, see 1. and 2. Schur lemma. On the other hand we have

$$M_{\mu\rho} = \sum_{g \in G} \sum_{r,s} D^i_{\mu r}(g)\, X_{rs}\, D^j_{s\rho}(g^{-1})$$

Let us choose $X_{rs} = \delta_{r\nu} \delta_{s\sigma}$ then

$$M_{\mu\rho} = \sum_{g \in G} D^i_{\mu\nu}(g)\, D^j_{\sigma\rho}(g^{-1}) = \lambda \delta_{\mu\rho} \delta_{ij}$$

Now we calculated λ for $i = j$ by setting $\mu = \rho$ and sum over μ.

$$\lambda\, d_i = \sum_g \sum_{\mu=1}^{d_i} D^i_{\mu\nu}(g)\, D^i_{\sigma\mu}(g^{-1}) = \sum_g \underbrace{D^i_{\sigma\nu}(\overbrace{g^{-1} g}^{=e})}_{\delta_{\sigma\nu}} = n \delta_{\sigma\nu}$$

Hence, $\lambda = \frac{n}{d_i} \delta_{\sigma\nu}$ and we conclude

$$\boxed{\frac{1}{n} \sum_{g \in G} D^i_{\mu\nu}(g) D^j_{\sigma\rho}(g^{-1}) = \frac{1}{d_i} \delta_{ij} \delta_{\mu\rho} \delta_{\nu\sigma}} \qquad \text{(I)}$$

Comments:

- (I) is valid for irreducible reps not necessarily unitary
- For unitary reps follows $D^j_{\sigma\rho}(g^{-1}) = D^{j\,*}_{\rho\sigma}(g)$ and proof is completed
- Extension to compact groups obvious

$$\frac{1}{n}\sum_g(\cdot) \Rightarrow \int_G \mathrm{d}g\,(\cdot)$$

- With proper interpretation also to non-compact uni-modular groups having $d_i = \infty$ for UIR.

Examples

- $G = U(1)$ then $D^m(g) = \mathrm{e}^{\mathrm{i}m\varphi}$, $d_m = 1$, $m \in \mathbb{Z}$, $g = g(\varphi)$
  $\int_{U(1)} \mathrm{d}g\, D^m(g) D^{n\,*}(g) = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \mathrm{e}^{\mathrm{i}m\varphi}\mathrm{e}^{-\mathrm{i}n\varphi} = \delta_{mn}$
- $G = T^1$ then $D^k(g) = \mathrm{e}^{-\mathrm{i}kx}$, $k \in \mathbb{R}$, $g = g(x)$, $x \in \mathbb{R}$ (see homework)
  $\int_{T^1} \mathrm{d}g\, D^k(g) D^{k'\,*}(g) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{2\pi} \mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{-k'x} = \delta(k-k')$

2.7.2 Abstract harmonic analysis

Consider finite group $G = \{g_1, g_2, \ldots, g_n\}$, $n = \operatorname{ord} G$ and a well-defined function

$$f: \begin{array}{l} G \to \mathbb{C} \\ g \mapsto f(g) \end{array}$$

Let

$$\vec{f} := (f(g_1), f(g_2), \ldots, f(g_n))$$

be element of vector space $V \simeq \mathbb{C}^n$ with scalar product

$$\langle \vec{h}, \vec{f} \rangle := \frac{1}{n} \sum_{n \in G} h^*(g) f(g)\,,$$

then for an arbitrary UIR

$$\vec{e}^{\,i}_{\mu\nu} := \sqrt{d_i}\left(D^i_{\mu\nu}(g_1), D^i_{\mu\nu}(g_2), \ldots, D^i_{\mu\nu}(g_n)\right)^{\mathrm{T}}$$

obeys

$$\langle \vec{e}^{\,j}_{\rho\sigma}, \vec{e}^{\,i}_{\mu\nu} \rangle = \frac{\sqrt{d_i d_j}}{n} \sum_{g \in G} D^{j\,*}_{\rho\sigma}(g) D^i_{\mu\nu}(g) = \delta_{ij}\delta_{\mu\rho}\delta_{\nu\rho}\,.$$

That is, it forms a (complete) orthonormal set in V.

Comment: For a fixed i there exist d_i^2 linearly independent unit vectors $\Rightarrow \sum_i d_i^2 \le n$. In other words, for finite groups there exist only a finite number of UIR.

Theorem of Burnside:

$$\sum_{\text{all UIR}} d_i^2 = n$$

Proof: Later

Conclusion: $\{\vec{e}^{\,i}_{\mu\nu}\}$ forms a complete set in $V \simeq \mathbb{C}^n$

$$\vec{f} = \sum_{i,\mu,\nu} d_i \left\langle \frac{1}{\sqrt{d_i}} \vec{e}^{\,i}_{\mu\nu}, \vec{f} \right\rangle \frac{1}{\sqrt{d_i}} \vec{e}^{\,i}_{\mu\nu}$$

with

$$\tilde{f}^i_{\nu\mu} := \left\langle \frac{1}{\sqrt{d_i}} \vec{e}^{\,i}_{\mu\nu}, \vec{f} \right\rangle = \frac{1}{n} \sum_g D^{i\,*}_{\mu\nu}(g) f(g) = \frac{1}{n} \sum_g D^i_{\nu\mu}(g^{-1}) f(g)\,.$$

Or for a fixed component $f(g)$ of $\vec{f}$

$$\boxed{\begin{aligned} f(g) &= \sum_{\text{all UIR } i} d_i \sum_{\mu,\nu=1}^{d_i} \tilde{f}^i_{\nu\mu}\, D^i_{\mu\nu}(g) \\ \tilde{f}^i_{\nu\mu} &= \frac{1}{n} \sum_{g\in G} f(g) D^i_{\nu\mu}(g^{-1}) \end{aligned}}$$

Above decomposition is called abstract Fourier or harmonic analysis.

Peter-Weyl-Theorem:

$$\boxed{\begin{aligned} f(g) &= \sum_{\text{all UIR } i} d_i \mathrm{Tr} \left(\tilde{f}^{\,i}\, D^i(g) \right) \\ \tilde{f}^{\,i} &= \frac{1}{n} \sum_{g\in G} f(g) D^i(g^{-1}) \end{aligned}}$$

Comments:

- Parseval equation (without proof)

$$\frac{1}{n} \sum_{g\in G} |f(g)|^2 = \sum_{\text{all UIR } i} \sum_{\mu,\nu=1}^{d_i} d_i |\tilde{f}^i_{\nu\mu}|^2$$

- Extension to compact groups and with proper interpretation even to uni-modular groups possible

Examples:

- $G = Z_2$: See Homework Problem 1
- $G = U(1)$: Fourier series

$$f(\varphi) = \sum_{m\in\mathbb{Z}} \tilde{f}^m \mathrm{e}^{-\mathrm{i}m\varphi}\,, \qquad \tilde{f}^m = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, f(\varphi) \mathrm{e}^{\mathrm{i}m\varphi}$$

- $G = T^1$: Fourier analysis

$$f(x) = \int_{\mathbb{R}} \mathrm{d}k\, \tilde{f}(k) \mathrm{e}^{-\mathrm{i}kx}\,, \qquad \tilde{f}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}x\, f(x) \mathrm{e}^{\mathrm{i}kx}$$

*** End of Lecture 2 ***