4. Solution to Homework in "Group Theory for Physicists" **SoSe 22**

Problem 11: Generators of $SO(3)$ in $L^2(\mathbb{R}^3)$

a) Use result of problem 10c)

$$\exp\{-\mathrm{i}\delta\alpha^a L_a\} = 1 + L_a^2(\cos\delta\alpha^a - 1) - \mathrm{i}L_a\sin\delta\alpha^a \approx 1 - \mathrm{i}L_a\delta\alpha^a + O((\delta\alpha)^2)$$

Hence

$$g(\vec{\delta\alpha}) \approx 1 - \mathrm{i}\sum_{a=1}^{3}\delta\alpha^a L_a = 1 + \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix}$$

b)

$$\delta\vec{x} = \vec{x}\,' - \vec{x} = \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2\delta\alpha^3 + x_3\delta\alpha^2 \\ x_1\delta\alpha^3 - x_3\delta\alpha^1 \\ -x_1\delta\alpha^2 + x_2\delta\alpha^1 \end{pmatrix} =: \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}$$

Hence

$$\left.\begin{aligned} X_1 &= -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha_1}\,\partial_k = x_3\partial_2 - x_2\partial_3 \\ X_2 &= -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha_2}\,\partial_k = x_1\partial_3 - x_3\partial_1 \\ X_3 &= -\sum_{k=1}^{3}\frac{\delta x_k}{\delta\alpha_3}\,\partial_k = x_2\partial_1 - x_1\partial_2 \end{aligned}\right\} \quad \vec{X} = -\vec{x}\times\vec{\nabla} = \frac{1}{\mathrm{i}\hbar}\,\vec{x}\times\vec{p} = \frac{1}{\mathrm{i}\hbar}\,\vec{L}$$

c) As $X_i = -\varepsilon_{ijk}x_j\partial_k$ (sum over repeated indices) $\Longrightarrow$

$$[X_i, X_j] = \varepsilon_{ilm}\varepsilon_{jkn}[x_l\partial_m, x_k\partial_n]$$

Consider

$$\begin{aligned} [x_l\partial_m, x_k\partial_n] &= x_l\partial_m x_k\partial_n - x_k\partial_n x_l\partial_m \\ &= x_l\left(\delta_{km} + \underline{x_k\partial_m}\right)\partial_n - x_k\left(\delta_{nl} + \underline{x_l\partial_n}\right)\partial_m \\ &= \delta_{km}x_l\partial_n - \delta_{nl}x_k\partial_m \end{aligned}$$

Hence

$$
\begin{aligned}
[X_i, X_j] &= \varepsilon_{ilm}\varepsilon_{jkn}\left(\delta_{km}x_l\partial_n - \delta_{nl}x_k\partial_m\right) \\
&= \varepsilon_{ilk}\varepsilon_{jkn}x_l\partial_n - \varepsilon_{ilm}\varepsilon_{jkl}x_k\partial_m \\
&\quad \text{use} \quad \varepsilon_{jkn} = -\varepsilon_{jnk} \quad \text{and} \quad \varepsilon_{ilm} = -\varepsilon_{iml} \\
&= \varepsilon_{iml}\varepsilon_{jkl}x_k\partial_m - \varepsilon_{ilk}\varepsilon_{jnk}x_l\partial_n \\
&\quad \text{use} \quad \varepsilon_{rst}\varepsilon_{uvt} = \left(\delta_{ru}\delta_{sv} - \delta_{rv}\delta_{su}\right) \\
&= \left(\underline{\delta_{ij}\delta_{mk}} - \delta_{ik}\delta_{mj}\right)x_k\partial_m - \left(\underline{\delta_{ij}\delta_{ln}} - \delta_{in}\delta_{lj}\right)x_l\partial_n \\
&= \delta_{in}\delta_{lj}x_l\partial_n - \delta_{ik}\delta_{mj}x_k\partial_m \\
&\quad \text{replace in 1. term} \quad n \to m\,, \quad l \to k \\
&= \underbrace{\left(\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm}\right)}_{\varepsilon_{ijr}\varepsilon_{mkr}} x_k\partial_m \\
&= -\varepsilon_{ijr}\varepsilon_{rkm}x_k\partial_m = \varepsilon_{ijr}X_r \qquad \Longrightarrow \qquad \boxed{{c_{ij}}^r = \varepsilon_{ijr}} \quad \#
\end{aligned}
$$

Cartan metric:

$$
\begin{aligned}
g_{kl} :=&\ {c_{kr}}^s{c_{ls}}^r = \varepsilon_{krs}\varepsilon_{lsr} = -\varepsilon_{krs}\varepsilon_{lrs} \\
=&\ -\left(\delta_{kl}\delta_{rr} - \delta_{kr}\delta_{rl}\right) \\
=&\ -\left(3\delta_{kl} - \delta_{kl}\right) = -2\delta_{kl} \qquad \#
\end{aligned}
$$

Metric tensors:

$$
(g_{kl}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \left(g^{kl}\right) = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}
$$

Casimir operator:

$$
C := g^{kl}X_kX_l = -\tfrac{1}{2}\,\vec{X}^{\,2} = \tfrac{1}{2\hbar}\,\vec{L}^{\,2}
$$

Problem 12: Generators of $SO(4)$ in $L^2(\mathbb{R}^4)$)

General considerations:

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, x_μ and ∂_μ denote coordinate and derivative, respectively. Consider operator

$$L_{\mu\nu} := x_\mu \partial_\nu - x_\nu \partial_\mu$$

Obviously, $L_{\mu\nu}$ is generator of rotation in $x_\mu - x_\nu$-plane

$\Longrightarrow$ set of all $L_{\mu\nu} = -L_{\nu\mu}$ generates rotations in $\mathbb{R}^n$

There exist $n(n-1)/2$ independent generators of $SO(n)$.

Derivation of the $so(n)$ algebra:

$$\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}] =& \left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\left(x_\rho\partial_\sigma - x_\sigma\partial_\rho\right) - \left(x_\rho\partial_\sigma - x_\sigma\partial_\rho\right)\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \\
=& \; x_\mu\partial_\nu x_\rho\partial_\sigma - x_\mu\partial_\nu x_\sigma\partial_\rho - x_\nu\partial_\mu x_\rho\partial_\sigma + x_\mu\partial_\nu x_\sigma\partial_\rho \\
& -x_\rho\partial_\sigma x_\mu\partial_\nu + x_\rho\partial_\sigma x_\nu\partial_\mu + x_\sigma\partial_\rho x_\mu\partial_\nu - x_\sigma\partial_\rho x_\nu\partial_\mu \\
=& \; x_\mu\left(\delta_{\nu\rho} + \underline{x_\rho\partial_\nu}\right)\partial_\sigma - x_\mu\left(\delta_{\nu\sigma} + \underline{x_\sigma\partial_\nu}\right)\partial_\rho - x_\nu\left(\delta_{\mu\rho} + \underline{x_\rho\partial_\mu}\right)\partial_\sigma + x_\nu\left(\delta_{\mu\sigma} + \underline{x_\sigma\partial_\mu}\right)\partial_\rho \\
& -x_\rho\left(\delta_{\mu\sigma} + \underline{x_\mu\partial_\sigma}\right)\partial_\nu + x_\rho\left(\delta_{\nu\sigma} + \underline{x_\nu\partial_\sigma}\right)\partial_\mu + x_\sigma\left(\delta_{\mu\rho} + \underline{x_\mu\partial_\rho}\right)\partial_\nu - x_\sigma\left(\delta_{\nu\rho} + \underline{x_\nu\partial_\rho}\right)\partial_\mu \\
=& \; \delta_{\nu\rho}\left(x_\mu\partial_\sigma - x_\sigma\partial_\mu\right) - \delta_{\nu\sigma}\left(x_\mu\partial_\rho - x_\rho\partial_\mu\right) - \delta_{\mu\rho}\left(x_\nu\partial_\sigma - x_\sigma\partial_\nu\right) + \delta_{\mu\sigma}\left(x_\nu\partial_\rho - x_\rho\partial_\nu\right) \\
=& \; \delta_{\nu\rho}L_{\mu\sigma} - \delta_{\nu\sigma}L_{\mu\rho} - \delta_{\mu\rho}L_{\nu\sigma} + \delta_{\mu\sigma}L_{\nu\rho}
\end{aligned}$$

a) Now $n = 4$ and $(x_1, x_2, x_3, x_4) = (x, y, z, t) \in \mathbb{R}^4$

We identify

$$\begin{aligned}
M_1 &= L_{32} & M_2 &= L_{13} & M_3 &= L_{21} \\
N_1 &= L_{14} & N_2 &= L_{24} & N_3 &= L_{34}
\end{aligned}$$

Now we can use above result of $so(n)$ algebra:

$$\left.\begin{aligned}
[M_1, M_2] &= [L_{32}, L_{13}] = L_{21} = M_3 \\
[M_2, M_3] &= [L_{13}, L_{21}] = L_{32} = M_1 \\
[M_3, M_1] &= [L_{21}, L_{32}] = L_{13} = M_2
\end{aligned}\right\} \qquad [M_i, M_j] = \varepsilon_{ijk} M_k$$

$$\left.\begin{aligned}
[M_1, N_1] &= [L_{32}, L_{14}] = 0 \\
[M_2, N_2] &= [L_{13}, L_{24}] = 0 \\
[M_3, N_2] &= [L_{21}, L_{34}] = 0
\end{aligned}\right\} \qquad [M_i, N_i] = 0$$

$$\left.\begin{aligned}
[M_1, N_2] &= [L_{32}, L_{24}] = L_{34} = N_3 \\
[M_2, N_3] &= [L_{13}, L_{34}] = L_{14} = N_1 \\
[M_3, N_1] &= [L_{21}, L_{14}] = L_{24} = N_2 \\
[M_1, N_3] &= [L_{32}, L_{34}] = L_{24} = -N_3 \\
[M_2, N_1] &= [L_{13}, L_{14}] = L_{34} = -N_1 \\
[M_3, N_2] &= [L_{21}, L_{24}] = L_{14} = -N_2
\end{aligned}\right\} \qquad [M_i, N_j] = \varepsilon_{ijk} N_k$$

$$[N_i, N_j] = [L_{i4}, L_{j4}] = -L_{ij} = L_{ji} = \varepsilon_{ijk} M_k$$

b) Let $J_k := \frac{1}{2}(M_k + N_k)$

$$\begin{aligned}
4[J_k, J_l] &= [M_k + N_k, M_l + N_l] \\
&= [M_k, M_l] + [M_k, N_l] + [N_k, M_l] + [N_k, N_l] \\
&= \varepsilon_{klm} M_m + \varepsilon_{klm} N_m - \varepsilon_{lkm} N_m + \varepsilon_{klm} M_m \\
&= \varepsilon_{klm}(2M_m + 2N_m) = 4\varepsilon_{klm} J_m
\end{aligned}$$

$$\Longrightarrow \qquad [J_k, J_l] = \varepsilon_{klm} J_m \qquad so(3) - \text{algebra}$$

Let $K_i := \frac{1}{2}(M_i - N_i)$

$$\begin{aligned}
4[K_i, K_j] &= [M_i - N_i, M_j - N_j] \\
&= [M_i, M_j] - [M_i, N_j] - [N_i, M_j] + [N_i, N_j] \\
&= \varepsilon_{ijk} M_k - \varepsilon_{ijk} N_k + \varepsilon_{jik} N_k + \varepsilon_{ijk} M_k \\
&= 2\varepsilon_{ijk}(M_k - N_k) = 4\varepsilon_{ijk} K_k
\end{aligned}$$

$$\Longrightarrow \qquad [K_i, K_j] = \varepsilon_{ijk} K_k \qquad so(3) - \text{algebra}$$

Finally we show the decoupling of both subalgebras

$$\begin{aligned}
[J_i, K_j] &= [M_i + N_i, M_j - N_j] \\
&= [M_i, M_j] - [M_i, N_j] + [N_i, M_j] - [N_i, N_j] \\
&= \varepsilon_{ijk} M_k - \varepsilon_{ijk} N_k - \varepsilon_{jik} N_k - \varepsilon_{ijk} M_k \\
&= 0
\end{aligned}$$

$$\Longrightarrow \qquad so(4) = so(3) \oplus so(3)$$

Casimir operators of $so(4)$

We have two Casimir from both $so(3)$ algebras:

$$\vec{J}^2 = J_1^2 + J_2^2 + J_3^2\,, \qquad \vec{K}^{\,2} = K_1^2 + K_2^2 + K_3^2$$

The corresponding labels of their UIR are $j, k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Hence, the UIR of $so(4)$ can be denoted by the pair (j, k)

Some relations

$$\begin{aligned} \vec{J}^2 =& \tfrac{1}{4}(\vec{M} + \vec{N})^2 = \tfrac{1}{4}(\vec{M}^{\,2} + \vec{M}\cdot\vec{N} + \vec{N}\cdot\vec{M} + \vec{N}^{\,2}) \\ & \quad\text{but}\quad \vec{M}\cdot\vec{N} = \vec{N}\cdot\vec{M} \quad\text{as}\quad [M_i, N_i] = 0 \\ =& \tfrac{1}{4}(\vec{M}^{\,2} + 2\vec{M}\cdot\vec{N} + \vec{N}^{\,2}) \end{aligned}$$

$$\vec{K}^{\,2} = \tfrac{1}{4}(\vec{M} - \vec{N})^2 = \tfrac{1}{4}(\vec{M}^{\,2} - 2\vec{M}\cdot\vec{N} + \vec{N}^{\,2})$$

$$\vec{J}^2 - \vec{K}^{\,2} = \vec{M}\cdot\vec{N}$$

$$\vec{J}^2 + \vec{K}^{\,2} = \tfrac{1}{4}(\vec{M}^{\,2} + \vec{N}^{\,2})$$