4. Solution to Homework in "Group Theory for Physicists" **SoSe 22**

Problem 9: Characters of $SO(3)$ and $SU(2)$

a) Let D be a representation of $SO(3) \simeq SU(2)$ then

$$D(R(\vec{\omega})) = D(R\mathrm{e}^{-\mathrm{i}\varphi J_3})D(R\mathrm{e}^{-\mathrm{i}\theta J_2})D(R\mathrm{e}^{-\mathrm{i}\omega J_3})D(R\mathrm{e}^{\mathrm{i}\theta J_2})D(R\mathrm{e}^{\mathrm{i}\varphi J_3})$$

and

$$\chi(R(\vec{\omega})) = \mathrm{Tr}\, D(R(\vec{\omega})) = \mathrm{Tr}\, \left[D(R\mathrm{e}^{-\mathrm{i}\omega J_3})\right] = \chi(\omega) \quad \#$$

b)

$$\begin{aligned}
\chi^j(\omega) &= \mathrm{Tr}\, D^j(R) = \sum_{m=-j}^{j} \langle jm|\mathrm{e}^{-\mathrm{i}\omega J_3}|jm\rangle \\
&= \sum_{m=-j}^{j} \mathrm{e}^{-\mathrm{i}\omega m} = \mathrm{e}^{-\mathrm{i}\omega j} \sum_{k=0}^{2j+1} (\mathrm{e}^{\mathrm{i}\omega})^k \qquad \text{geom. series} \\
&= \mathrm{e}^{-\mathrm{i}\omega j} \frac{1-\mathrm{e}^{\mathrm{i}\omega(2j+1)}}{1-\mathrm{e}^{\mathrm{i}\omega}} = \frac{\mathrm{e}^{-\mathrm{i}\omega(j+\frac{1}{2})} - \mathrm{e}^{\mathrm{i}\omega(j+\frac{1}{2})}}{\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega} - \mathrm{e}^{\frac{\mathrm{i}}{2}\omega}} \\
&= \frac{\sin(j+\frac{1}{2})\omega}{\sin\frac{\omega}{2}} \quad \#
\end{aligned}$$

c)

$$\begin{aligned}
\chi^{j_1}(\omega)\chi^{j_2}(\omega) &= \frac{\mathrm{e}^{-\mathrm{i}\omega(j_1+\frac{1}{2})} - \mathrm{e}^{\mathrm{i}\omega(j_1+\frac{1}{2})}}{2\mathrm{i}\sin\frac{\omega}{2}} \sum_{m=-j_2}^{j_2} \mathrm{e}^{\mathrm{i}\omega m} \\
&\quad \text{let} \quad j_1 \geq j_2 \quad \text{without loss of generality} \\
&= \frac{1}{2\mathrm{i}\sin\frac{\omega}{2}} \sum_{m=-j_2}^{j_2} \left(\mathrm{e}^{\mathrm{i}\omega(j_1+m+\frac{1}{2})} - \mathrm{e}^{-\mathrm{i}\omega(j_1+m+\frac{1}{2})}\right) \\
&\quad \text{set} \quad j = j_1 + m \\
&= \frac{1}{2\mathrm{i}\sin\frac{\omega}{2}} \sum_{j=j_1-j_2}^{j_1+j_2} \left(\mathrm{e}^{\mathrm{i}\omega(j+\frac{1}{2})} - \mathrm{e}^{-\mathrm{i}\omega(j+\frac{1}{2})}\right) \\
&= \sum_{j=j_1-j_2}^{j_1+j_2} \frac{\sin(j+\frac{1}{2})\omega}{\sin\frac{\omega}{2}} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \chi^j(\omega) \\
&\quad \text{as LHS is symmetric in} \quad j_1 \leftrightarrow j_2
\end{aligned}$$

Hence the addition of two angular momenta representation spaces is given by the reduction

$$D^{j_1} \otimes D^{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^j$$

Problem 10: Generators of $SO(3)$ for $j = 1$ (3-dim. representation)

Explicit calculations result in

$$L_1 L_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_2 L_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_2 L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad L_3 L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_3 L_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L_1 L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$L_1^2 = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \qquad L_2^2 = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \qquad L_3^2 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

a)

$$\left.\begin{aligned} [L_1, L_2] &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{i}L_3 \\ [L_2, L_3] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \mathrm{i}L_1 \\ [L_3, L_1] &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathrm{i}L_2 \end{aligned}\right\} \qquad [L_k, L_m] = \mathrm{i}\varepsilon_{kmn} L_n$$

b) L_k^2 is unit matrix in subspace orthogonal to the k-axis and vanishes on the one-dim. subspace spanned by the k-axis

Hence its matrix elements are given by

$$\left(L_k^2\right)_{mn} = \delta_{mn} - \delta_{km}\delta_{kn}$$

Obviously

$$L_k^0 = 1 \qquad \text{and} \qquad L_k^{2m} = L_k^2 \qquad \text{for} \qquad m = 1, 2, 3, \ldots$$

and

$$L_k^{2m+1} = L_k \qquad \text{for} \qquad m = 0, 1, 2, 3, \ldots$$

c)

$$\begin{aligned}
\exp\{-\mathrm{i}\varphi L_k\} &= \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathrm{i}\varphi L_k)^n = \sum_{m=0}^{\infty} \frac{1}{(2m)!} (-\mathrm{i}\varphi L_k)^{2m} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} (-\mathrm{i}\varphi L_k)^{2m+1} \\
&= \underbrace{1 - L_k^2}_{m=0 \text{ term}} + L_k^2 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \varphi^{2m}}_{\cos\varphi} + (-\mathrm{i}L_k) \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \varphi^{2m+1}}_{\sin\varphi} \\
&= 1 + L_k^2(\cos\varphi - 1) - \mathrm{i}L_k \sin\varphi \qquad \#
\end{aligned}$$

Explicitly

$$\exp\{-\mathrm{i}\varphi L_1\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & 0 \\ 0 & 0 & \cos\varphi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sin\varphi \\ 0 & -\sin\varphi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\exp\{-\mathrm{i}\varphi L_2\} = \begin{pmatrix} \cos\varphi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\varphi \end{pmatrix} - \begin{pmatrix} 0 & 0 & -\sin\varphi \\ 0 & 0 & 0 \\ \sin\varphi & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

$$\exp\{-\mathrm{i}\varphi L_3\} = \begin{pmatrix} \cos\varphi & 0 & 0 \\ 0 & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \sin\varphi & 0 \\ -\sin\varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Tr}\,\left[\mathrm{e}^{-\mathrm{i}\varphi L_k}\right] = \mathrm{Tr}\,\left[1 + L_k^2(\cos\varphi - 1) - \mathrm{i}L_k \sin\varphi\right] = 3 + 2(\cos\varphi - 1) = 1 + 2\cos\varphi$$

$$\chi^1(\varphi) = \frac{\sin\frac{3}{2}\varphi}{\sin\frac{\varphi}{2}} = \frac{3\sin\frac{\varphi}{2} - 4\sin^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} = 3 - 4\sin^2\tfrac{\varphi}{2} = 1 + 2(1 - \sin^2\tfrac{\varphi}{2}) = 1 + 2\cos\varphi \quad \#$$