2. Solution to Homework in "Group Theory for Physicists" **SoSe 22**

Problem 3: One-dim. UIR of the Braid group B_n

Generators obey:

$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i \quad \text{for} \quad |i-j|>1$$
$$\varepsilon_i\varepsilon_{i+1}\varepsilon_i = \varepsilon_{i+1}\varepsilon_i\varepsilon_{i+1}$$

a) Make ansatz for 1-D UIR for generators: $D^\alpha(\varepsilon_j) = \mathrm{e}^{\mathrm{i}\alpha_j}$

- Unitarity: $\alpha_j \in [0, 2\pi[$
- First property: $D^\alpha(\varepsilon_i\varepsilon_j) = \mathrm{e}^{\mathrm{i}(\alpha_i+\alpha_j)} = D^\alpha(\varepsilon_j\varepsilon_i)$
- Second property: $D^\alpha(\varepsilon_i\varepsilon_{i+1}\varepsilon_i) = \mathrm{e}^{\mathrm{i}(\alpha_i+\alpha_{i+1}+\alpha_i)} \overset{!}{=} D^\alpha(\varepsilon_{i+1}\varepsilon_i\varepsilon_{i+1}) = \mathrm{e}^{\mathrm{i}(\alpha_{i+1}+\alpha_i+\alpha_{i+1})}$
  $\implies \alpha_{i+1} = \alpha_i \bmod 2\pi \implies \alpha_{i+1} = \alpha_i$ for all i.

Hence $\alpha \in [0, 2\pi[$ characterises the full set of all 1-dim. UIR of B_n

b) Restriction to S_n implies a third property $\varepsilon_i^2 = e$

$$\implies D^\alpha(\varepsilon_i^2) = \mathrm{e}^{2\mathrm{i}\alpha} \overset{!}{=} 1 \implies \alpha = \begin{cases} 0 & \text{trivial reps} \\ \pi & \text{anti-sym. reps} \end{cases}$$

Comments:

Consider a quantum system of n identical particles characterised by the n-particle wave function $\Psi(x_1,\ldots,x_n)$. The permutation of these particles may be represented by a permutation of the positions x_i: $P = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$. The representation of such a permutation in the n-particle Hilbert space $L^2(\mathbb{R}^{3n})$ is given by

$$D^\alpha(P)\Psi(x_1,\ldots,x_n) := \Psi(x_{\pi_1},\ldots,x_{\pi_n})$$

For identical particles the physics should not change, that is, $|\Psi(x_1,\ldots,x_n)|^2 = |\Psi(x_{\pi_1},\ldots,x_{\pi_n})|^2$. Hence, D^α must be a 1-dim. UIR of S_n:

- Trivial reps.: $D^0(P)\Psi(x_1,\ldots,x_n) := \Psi(x_1,\ldots,x_n) = \Psi(x_{\pi_1},\ldots,x_{\pi_n})$. **Bosons**
- Anti-sym. reps.: $D^\pi(P)\Psi(x_1,\ldots,x_n) := (-1)^k\Psi(x_1,\ldots,x_n)$. **Fermions**
  $k =$ number of transposition in P; odd permuations pick up a minus sign.

For elementary particles these are the only physical 1D UIR of B_n. However, for "quasi particles" (collective excitation in a solid) any α may be realized. These are called **Anyons**. See, for example,

J. Jacak, R. Gonczarek, L. Jacak and I. Jóźwiak, *Application of Braid Groups in 2D Hall System Physics* (World Scientific, Singapore, 2012) https://doi.org/10.1142/8512

Problem 4: One-dim. UIR of $SO(2) \simeq U(1)$

Defining representation of $SO(2)$ in $\mathbb{R}^2$

$$g(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \qquad \varphi \in [0, 2\pi[.$$

a) Consider the homomorphism

$$H: \begin{array}{ccc} SO(2) & \to & U(1) \\ g(\varphi) & \mapsto & \mathrm{e}^{\mathrm{i}\varphi} \end{array}$$

Note that using the trigonometric addition theorems we have

$$g(\varphi_1)g(\varphi_2) = \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix} = \begin{pmatrix} \cos(\varphi_1+\varphi_2) & -\sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \end{pmatrix}$$

Hence, the group law is mapped under H as $g(\varphi_1)g(\varphi_2) = g(\varphi_1+\varphi_2) \mapsto \mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}$.
Obviously this is invertable and H is actually an isomorphism. $SO(2) \simeq U(1)$.
Alternatively, let $\vec{x} = (x_1, x_2)^T \in \mathbb{R}^2$, then we can map it onto $z := x_1 + \mathrm{i}x_2 \in \mathbb{C}$. Hence the rotation $g(\varphi)$ in $\mathbb{R}^2$ is replaced by the multiplication of z with phase $\mathrm{e}^{\mathrm{i}\varphi}$.
Consider unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}, \qquad U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = U^{-1},$$

then the 2-dim. reducible matrix reps. is reduced to

$$U^\dagger g(\varphi) U = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix}.$$

b) Ansatz for 1-dim. UIR: $D^\alpha(g) = \mathrm{e}^{\mathrm{i}\alpha(\varphi)}$ with $\alpha(\varphi) \in [0, 2\pi]$.

- $g(\varphi_1)g(\varphi_1) = g(\varphi_1+\varphi_2) \Longrightarrow \alpha(\varphi_1) + \alpha(\varphi_2) = \alpha(\varphi_1+\varphi_2) \bmod 2\pi$.
  Hence α is linear in φ.
- $g(0) = \mathbf{1} = g(2\pi) \Longrightarrow \alpha(0) = 2\pi m$ and $\alpha(2\pi) = 2\pi n$ with $m, n \in \mathbb{Z}$

Conclusion: $\alpha(\varphi) = m \cdot \varphi$ and

$$\boxed{D^m(g) = \mathrm{e}^{\mathrm{i}m\varphi} \quad \text{with} \quad m \in \mathbb{Z}}$$

is 1-dim. UIR of $SO(2) \simeq U(1)$.

Problem 5: Translations in $\mathbb{R}^3$ and its 1D UIR

$$T^3 : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ \vec{a} \mapsto \vec{a} + \vec{x} \end{cases} \qquad \text{with group element} \qquad g(\vec{x}) = \begin{pmatrix} \mathbf{1}_3 & \vec{x} \\ \vec{0}^T & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{x} \in \mathbb{R}^3$$

a) Neutral element $g(\vec{0}) = \mathbf{1}_4$ obvious.

Hence we only need to verify the group law of translations, which implies $g(\vec{x})g(\vec{y}) = g(\vec{x}+\vec{y})$:

$$g(\vec{x})g(\vec{y}) = \begin{pmatrix} \mathbf{1}_3 & \vec{x} \\ \vec{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_3 & \vec{y} \\ \vec{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_3 & \vec{x} + \vec{y} \\ \vec{0}^T & 1 \end{pmatrix} = g(\vec{x} + \vec{y})$$

Obviously the inverse of $g(\vec{x})$ is given by $g(-\vec{x})$ and T^3 is abelian.

However,

$$g^{-1}(\vec{x}) = g(-\vec{x}) = \begin{pmatrix} \mathbf{1}_3 & -\vec{x} \\ \vec{0}^T & 1 \end{pmatrix} \neq g^\dagger(\vec{x}) = \begin{pmatrix} \mathbf{1}_3 & \vec{0} \\ \vec{x}^T & 1 \end{pmatrix} \notin T^3$$

The above matrix representation of T^3 is neither unitary nor irreducible. It acts on $\mathbb{R}^4$ with $(0, 0, 0, 1)^T$ spanning an invariant subspace.

b) Ansatz $D_{\vec{k}}(g(\vec{x})) = \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}$ is representation as

$$D_{\vec{k}}(g(\vec{0})) = 1 \qquad \text{and} \qquad D_{\vec{k}}(g(\vec{x} + \vec{y})) = D_{\vec{k}}(g(\vec{x}))D_{\vec{k}}(g(\vec{y}))$$

Unitarity follows from $D_{\vec{k}}(g(-\vec{x})) = D^*_{\vec{k}}(g(\vec{x}))$