1. Solution to Homework in "Group Theory for Physicists" SoSe 22

Problem 1: Cayley's theorem for $Z_2 = \{-1, 1\}$

Let $f(\pm 1) \in \mathbb{C}$ with $|f(\pm 1)| < \infty$ be well defined. Then for any $\sigma_0 \in Z_2$ with $\circ$ being the usual multiplication of numbers

$$\sum_{\sigma \in Z_2} f(\sigma\sigma_0) = f(-\sigma_0) + f(\sigma_0) = f(-1) + f(1) = \sum_{\sigma \in Z_2} f(\sigma)\,.$$

Hence Cayley's rearrangement theorem is valid, that is,

$$\sum_{\sigma \in Z_2} f(\sigma\sigma_0) = \sum_{\sigma \in Z_2} f(\sigma), \quad \forall \sigma_0 \in Z_2.$$

Special cases:

- $f(\sigma) = \sigma$

$$\sum_{\sigma \in Z_2} \sigma = -1 + 1 = 0$$

- $f(\sigma) = \sigma^2$

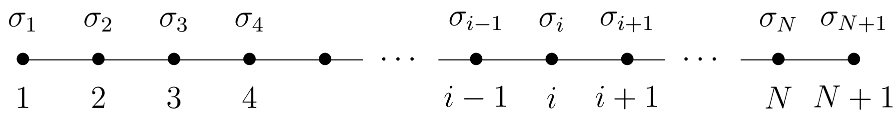
$$\sum_{\sigma \in Z_2} \sigma^2 = \sum_{\sigma \in Z_2} 1 = 2$$

- For Ising model

$$\begin{aligned} \cosh(\alpha) + \sigma \sinh(\alpha) &= \frac{1}{2}\left(\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}\right) + \frac{\sigma}{2}\left(\mathrm{e}^{\alpha} - \mathrm{e}^{-\alpha}\right) \\ &= \frac{1+\sigma}{2}\,\mathrm{e}^{\alpha} + \frac{1-\sigma}{2}\,\mathrm{e}^{-\alpha} \\ &= \left\{ \begin{array}{lcl} \mathrm{e}^{\alpha} & \text{for} & \sigma = 1 \\ \mathrm{e}^{-\alpha} & & \sigma = -1 \end{array} \right\} = \mathrm{e}^{\alpha\sigma} \end{aligned}$$

Problem 2: The one-dimensional Ising model

One-dimensional lattice with $N+1$ Ising-"Spins" attached, $\sigma_i \in Z_2$, $i = 1, 2, 3, \ldots, N+1$:



Hamiltonian:

$$H := -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}\,, \quad J \in \mathbb{R}.$$

Consider:

$$\mathrm{e}^{-\beta H} = \exp\left\{ \beta J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} \right\} = \prod_{i=1}^{N} \mathrm{e}^{\beta J \sigma_i \sigma_{i+1}} = \prod_{i=1}^{N} \Big(\cosh(\beta J) + \sigma_i \sigma_{i+1} \sinh(\beta J) \Big)$$

Partition function:

$$\begin{aligned} Z(\beta) &= \sum_{\sigma_1 \in Z_2} \cdots \sum_{\sigma_{N+1} \in Z_2} \prod_{i=1}^{N} \cosh(\beta J) \Big(1 + \lambda(\beta) \sigma_i \sigma_{i+1} \Big) \quad \text{with} \quad \lambda(\beta) := \tanh(\beta J) \\ &= \cosh^N(\beta J) \left[\sum_{\sigma_1 \in Z_2} \cdots \sum_{\sigma_{N+1} \in Z_2} \Big(1 + \lambda \sigma_1 \sigma_2\Big)\Big(1 + \lambda \sigma_2 \sigma_3\Big) \cdots \Big(1 + \lambda \sigma_N \sigma_{N+1}\Big) \right] \end{aligned}$$

$$\begin{aligned} &\text{1. sum :} && \sum_{\sigma_1 \in Z_2} \Big(1 + \lambda \sigma_1 \sigma_2\Big) = 2 \\ &\text{2. sum :} && \sum_{\sigma_2 \in Z_2} \Big(1 + \lambda \sigma_2 \sigma_3\Big) = 2 \\ & && \vdots \\ &N\text{−th sum :} && \sum_{\sigma_N \in Z_2} \Big(1 + \lambda \sigma_N \sigma_{N+1}\Big) = 2 \\ &(N+1)\text{−th sum :} && \sum_{\sigma_{N+1} \in Z_2} 1 = 2 \end{aligned}$$

Partition function for free boundary conditions:

$$\boxed{Z(\beta) = 2^{N+1} \cosh^N(\beta J)}$$

Free energy in thermodynamic limit:

$$F(\beta) := -\frac{1}{\beta} \lim_{N \to \infty} \frac{1}{N} \ln Z(\beta) = -\frac{1}{\beta} \ln\Big(2 \cosh \beta J \Big)$$

Alternative way of doing the sum

Consider:

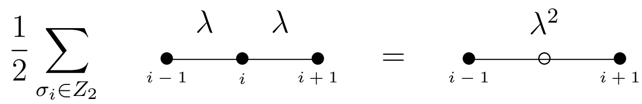
$$\sum_{\sigma_i \in Z_2} \Big(1 + \lambda \sigma_{i-1}\sigma_i\Big)\Big(1 + \lambda \sigma_i \sigma_{i+1}\Big) =$$

$$= \sum_{\sigma_i \in Z_2} \Big(1 + \lambda\sigma_{i-1}\sigma_i + \lambda\sigma_i\sigma_{i+1} + \lambda^2 \sigma_{i-1}\sigma_i^2\sigma_{i+1}\Big)$$

$$= \quad 2 + \quad 0 \quad + \quad 0 \quad + 2\lambda^2\sigma_{i-1}\sigma_{i+1}$$

$$= 2\Big(1 + \lambda^2 \sigma_{i-1}\sigma_{i+1}\Big)$$

Symbolically:



Doing the $N-1$ intermediate sums gives

$$Z(\beta) = 2^{N-1}\cosh^N(\beta J) \sum_{\sigma_1 \in Z_2} \sum_{\sigma_{N+1} \in Z_2} \Big(1 + \lambda^N \sigma_1 \sigma_{N+1}\Big)$$

For open boundary condition we recover the result on pervious page

For periodic boundary conditions we have $\sigma_{N+1} = \sigma_1$

$$Z_{PB}(\beta) = 2^{N-1}\cosh^N(\beta J) \sum_{\sigma_1 \in Z_2} \Big(1 + \lambda^N\Big) = 2^N \cosh^N(\beta J)\Big(1 + \lambda^N\Big)$$

and

$$F_{PB}(\beta) := -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \ln Z_{PB}(\beta) = -\frac{1}{\beta}\ln\Big(2\cosh\beta J\Big) = F(\beta)$$

Note that $\lambda(\beta) = \tanh \beta J$ and therefore $|\lambda(\beta)| < 1$ for all $\beta < \infty$.

Supplement:

Normalized measure on Z_2 and group average:

$$\langle \cdot \rangle_{Z_2} := \frac{1}{2} \sum_{\sigma \in Z_2} (\cdot) \qquad \text{Haar measure}$$

Harmonic analysis on Z_2:

Z_2 has 2 UIR:

$$D^0(\sigma) = 1 \ \text{ trivial representation}$$

$$D^1(\sigma) = \sigma \ \text{ faithful representation}$$

Let

$$f_0 := \frac{1}{2} \sum_{\sigma \in Z_2} f(\sigma) D^0(\sigma) \,, \qquad f_1 := \frac{1}{2} \sum_{\sigma \in Z_2} f(\sigma) D^1(\sigma)$$

be the Fourier coefficients of the harmonic analysis for function $f(\sigma)$. Then

$$f(\sigma) = \sum_{i=0}^{1} f_i \, D^i(\sigma)$$

Our example $f(\sigma) = \mathrm{e}^{\alpha\sigma}$ results in

$$f_0 = \cosh\alpha \,, \qquad f_1 = \sinh\alpha$$

Hence

$$f(\sigma) = \mathrm{e}^{\alpha\sigma} = \cosh\alpha \, D^0(\sigma) + \sinh\alpha \, D^1(\sigma) = \cosh\alpha + \sigma \sinh\alpha$$

Our λ is in essence the Fourier coefficient of the Boltzmann factor of the next-neighbour interaction for the non-trivial UIR divided by that for the trivial UIR.