Problem 11: Generators of $SO(3)$ in $L^2(\mathbb{R}^3)$.

Let L_1, L_2 and L_3 be the generators of rotations in $\mathbb{R}^3$ as defined in Problem 10 generating finite rotations

$$g(\vec{\alpha}) := \exp\left\{-\mathrm{i}\sum_{a=1}^{3}\alpha^a L_a\right\} \qquad \vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$$

a) Show that for small angles $\delta\alpha^a$ the rotation matrix explicitly reads

$$g(\delta\vec{\alpha}) = 1 + \begin{pmatrix} 0 & -\delta\alpha^3 & \delta\alpha^2 \\ \delta\alpha^3 & 0 & -\delta\alpha^1 \\ -\delta\alpha^2 & \delta\alpha^1 & 0 \end{pmatrix} + O(\delta\vec{\alpha}^2)$$

b) Consider now the infinitesimal rotation of an arbitrary point $\vec{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$. That is $\delta\vec{x} := \vec{x}\,' - \vec{x}$ with $\vec{x}\,' := g(\delta\vec{\alpha})\vec{x}$. Calculate the generators defined by

$$X_a := -\sum_{k=1}^{3} U_{ak}(\vec{x})\frac{\partial}{\partial x_k} \quad \text{with} \quad U_{ak}(\vec{x}) := \frac{\delta x_k}{\delta\alpha^a}.$$

Show that these generators are related to the angular momentum operator $\vec{L} := \vec{x} \times \vec{p}$ acting on $L^2(\mathbb{R}^3)$.

c) Calculate the structural constants for the algebra formed by X_1, X_2, X_3 and the associated Cartan metric and Casimir operator.

Problem 12: Generators of $SO(4)$ in $L^2(\mathbb{R}^4)$.

Consider the following operators acting on $L^2(\mathbb{R}^4)$ with $(x, y, z, t) \in \mathbb{R}^4$:

$$M_1 := z\partial_y - y\partial_z\,, \quad M_2 := x\partial_z - z\partial_x\,, \quad M_3 := y\partial_x - x\partial_y\,,$$
$$N_1 := x\partial_t - t\partial_x\,, \quad N_2 := y\partial_t - t\partial_y\,, \quad N_3 := z\partial_t - t\partial_z\,.$$

a) Show that these operators obey the $so(4)$ algebra

$$[M_i, M_j] = \varepsilon_{ijk}M_k\,, \quad [M_i, N_i] = 0\,, \quad [M_i, N_j] = \varepsilon_{ijk}N_k\,, \quad [N_i, N_j] = \varepsilon_{ijk}M_k\,.$$

b) Consider a new basis of this algebra defined by

$$J_i := \frac{M_i + N_i}{2}\,, \quad K_i := \frac{M_i - N_i}{2}$$

and show that the operators (J_1, J_2, J_3) and (K_1, K_2, K_3) separately close a $so(3)$ algebra, that is $so(4) = so(3) \oplus so(3)$.