4. Homework in "Group Theory for Physicists" **SoSe 22**

Problem 9: Characters of $SO(3)$ and $SU(2)$.

a) From Exercise 10 we know that an arbitrary element of $SO(3)$ and $SU(2)$ can be expressed in terms of

$$R(\vec{\omega}) = \mathrm{e}^{-\mathrm{i}\varphi J_3}\mathrm{e}^{-\mathrm{i}\theta J_2}\mathrm{e}^{-\mathrm{i}\omega J_3}\mathrm{e}^{\mathrm{i}\theta J_2}\mathrm{e}^{\mathrm{i}\varphi J_3}\,, \quad \vec{\omega} = (\omega\sin\theta\cos\varphi, \omega\sin\theta\sin\varphi, \omega\cos\theta)^T.$$

Show that for any representation its character only depends on the rotation angle ω.

b) Show that for the UIR D^j in $\mathbb{C}^{2j+1}$, i.e. $d_j = 2j+1$, the character explicitly reads

$$\chi^j(\omega) = \frac{\sin(j+\frac{1}{2})\omega}{\sin\frac{\omega}{2}}.$$

Hint: $\mathbb{C}^{2j+1} = \mathrm{span}\{|jm\rangle\}$, where $J_3\,|jm\rangle = m\,|jm\rangle\,,\ m = -j, -j+1, \ldots, j-1, j.$

c) Show that

$$\chi^{j_1}(\omega)\chi^{j_2}(\omega) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \chi^j(\omega).$$

Problem 10: Generators of $SO(3)$

Let L_k ($k \in \{1,2,3\}$) be 3×3 matrices whose elements are defined by $(L_k)_{ij} := -\mathrm{i}\varepsilon_{ijk}$, where ε_{ijk} is the total anti-symmetric tensor in 3 dimension. That is,

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Show that these matrices fulfill the relation $[L_k, L_m] = \mathrm{i}\varepsilon_{kmn}L_n$.

b) Show that $(L_k^2)_{mn} = \delta_{mn} - \delta_{km}\delta_{kn}$ and therefore $L_k^{2r} = L_k^2$ for $r = 1, 2, 3, \ldots$.

c) Show that $\exp\{-\mathrm{i}\varphi L_k\} = 1 + L_k^2(\cos\varphi - 1) - \mathrm{i}L_k\sin\varphi$
and calculate the explicit 3×3 matrices. Proof $\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}\varphi L_k} = \chi^1(\varphi)$.