**3. Homework in "Group Theory for Physicists"** **SoSe 22**

**Problem 6:** Classes of the dihedral group $D_n$.

a) Let $s$ and $d$ be the generators of $D_n$ obeying $d^n = e = s^2$ and $sd = d^{-1}s = d^{n-1}s$.

Show that $D_n$ has $\frac{n}{2} + 3$ classes for even $n$ and $\frac{n-1}{2} + 2$ classes for odd $n$.

Hint: Use relation $sd^k s = d^{-k} = d^{n-k}$ and construct classes with pure rotations first.

b) Construct the Character table for $D_4$.

**Problem 7:** Orthonormal projection operators for finite groups.

Let $\{j\}$ be the complete set of all UIR of a finite group $G$ with dimension $d_j$, $n = \operatorname{ord} G$ and $\chi^j(g)$ be the character associated with the UIR $j$ for group element $g \in G$. Show that for an arbitrary in general reducible but unitary representation $D$ of $G$ the operator

$$\mathbb{E}^j := \frac{d_j}{n} \sum_{g \in G} \chi^{j\,*}(g) D(g)$$

projects onto the unitary irreducible subspaces $\Sigma^j$ of $D$ associated with the UIR $D^j$.

Proof the properties $\mathbb{E}^{j\,\dagger} = \mathbb{E}^j$, $\mathbb{E}^j \mathbb{E}^k = \mathbb{E}^j \delta_{jk}$ and $\sum_{\{j\}} \mathbb{E}^j = 1$.

**Problem 8:** The dynamical $SO(4)$ symmetry of the Kepler-Problem

Consider the well-known 2-body Kepler problem describing the motion of a planet with (reduced) mass $m$ around a central star of (total) mass $M$. Let $E < 0$ be the total energy of the effective bounded one-body motion characterized by

$$\frac{m}{2}\,\dot{\vec{r}}^2 - G\frac{Mm}{r} = E\,, \quad r = |\vec{r}|\,, \quad G \text{ is the gravitational constant}\,.$$

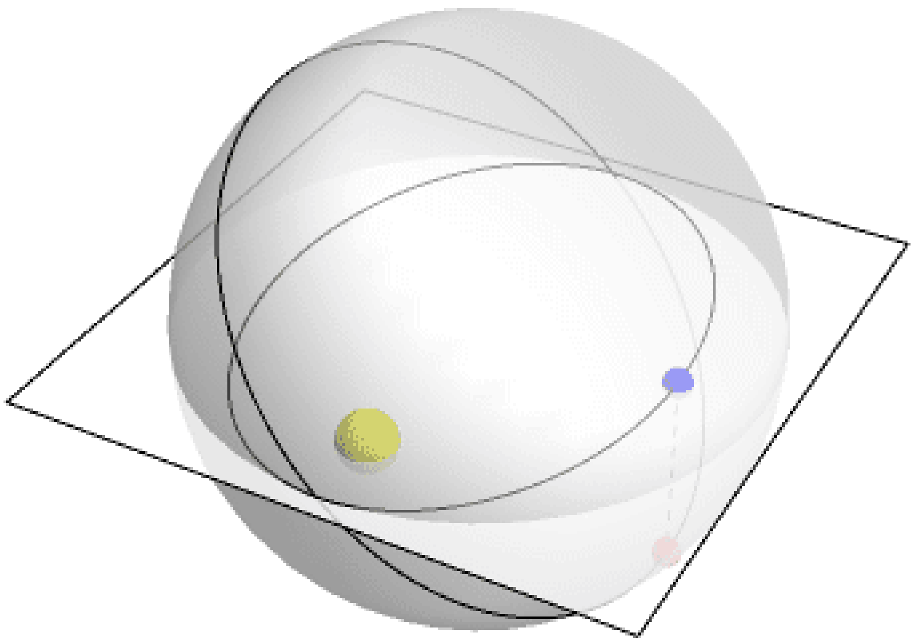
Consider now a new "time" $s$ along the Kepler orbit defined by $\mathrm{d}s = \mathrm{d}t/r$, that is, $\mathrm{d}t/\mathrm{d}s = r =: t'$. The prime denotes the derivative with respect to the new time $s$. The time $t$ will now be considered as a fourth independent degree of freedom coupled to the others via its equation of motion $t' = r$.

a) Show that the above equation for the Kepler orbit reads in the new time

$$(\vec{r}\,')^2 - 2GMt' = \frac{2E}{m}\,(t')^2\,.$$

b) Define the normalised 4-velocity $u := (u_0, \vec{u}) \in \mathbb{R}^4$ where $u_0 := t'/r_E$ and $\vec{u} := \vec{r}\,'/v_E r_E$ with $v_E^2 := 2m/|E|$ and $r_E := GM/v_E^2$. Show that the orbit of the 4-velocity is described by unit circles in $\mathbb{R}^4$ with center $(1, \vec{0})$.

c) For $E = 0$ show that the components of the 4-velocity $u_0 := t'/r_S$ and $\vec{u} := \vec{r}\,'/r_S c$, where $r_S := 2GM/c^2$, obey the relation $\vec{u}^{\,2} = u_0$.



Click on above picture to run the animation!