2. Homework in "Group Theory for Physicists" **SoSe 22**

One-dimension Unitary Irreducible Representations (UIR)

Problem 3: The braid group B_n is defined via its generators ε_i, $i = 1, \ldots, n-1$, obeying

$$\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad \text{for} \quad |i-j| > 1$$
$$\varepsilon_i \varepsilon_{i+1} \varepsilon_i = \varepsilon_{i+1} \varepsilon_i \varepsilon_{i+1}$$

a) Show that the complete set of one-dimensional UIR of B_n can be enumerated by the uncountable infinite set $\alpha \in [0, 2\pi[$.

b) The generators P_i of the permutation group S_n can be defined via above generators $P_i = \varepsilon_i$ with the additional requirement $P_i^2 = e$, where e is the trivial group element. Show that the one-dimensional UIR of S_n are fully characterizable by two distinct values of α in above set.

Problem 4: Consider the group $SO(2)$ of proper rotations in the 2-dimensional plane with group elements

$$g(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \qquad \varphi \in [0, 2\pi[.$$

a) Show that $SO(2)$ is isomorphic to $U(1)$, the group of unitary rotations in the complex plane $\{\mathrm{e}^{\mathrm{i}\varphi} | \varphi \in [0, 2\pi[\}$.

b) Show that all 1-dimensional UIR of $U(1) \simeq SO(2)$ are characterizable by a countable infinite set $m \in \mathbb{Z}$.

Problem 5: Consider the group T^3 of linear translations in $\mathbb{R}^3$. That is, the group of translations moving an arbitrary fixed $\vec{a} \in \mathbb{R}^3$ into $\vec{a} + \vec{x} \in \mathbb{R}^3$:

$$T^3 : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ \vec{a} \mapsto \vec{a} + \vec{x} \end{cases} \qquad \vec{x} \in \mathbb{R}^3.$$

a) Show that the elements of T^3 can be represented by a 4×4 matrix of the form

$$g(\vec{x}) = \begin{pmatrix} \mathbf{1}_3 & \vec{x} \\ \vec{0}^T & 1 \end{pmatrix}, \text{ where } \mathbf{1}_3 \text{ is the 3d unit matrix, } \vec{0}^T := (0,0,0) \text{ and } \vec{x} \in \mathbb{R}^3.$$

Is this representation unitary and/or irreducible?

b) Show that the 1-dimensional UIR of T^3 are given by $D_{\vec{k}}(g) := \exp\{-\mathrm{i}\vec{k} \cdot \vec{x}\}$, $\vec{k} \in \mathbb{R}^3$.