1. Homework in "Group Theory for Physicists" **SoSe 22**

Let $Z_2 := \{\sigma \,|\sigma = \pm 1\}$ form a group of order 2 with group composition law being the usual multiplication of its elements.

Problem 1: Cayley's theorem

Show that for arbitrary well-behaved finite values $f(\pm 1) \in \mathbb{C}$, i.e. $|f(\pm 1)| < \infty$,

$$f : \begin{array}{rcl} Z_2 & \to & \{f(1), f(-1)\} \\ \sigma & \mapsto & f(\sigma) \end{array} ,$$

Cayley's theorem (rearrangement theorem) is valid, that is,

$$\sum_{\sigma \in Z_2} f(\sigma\sigma_0) = \sum_{\sigma \in Z_2} f(\sigma), \quad \forall \sigma_0 \in Z_2.$$

Calculate the sum for $f(\sigma) = \sigma$ and $f(\sigma) = \sigma^2$ and show that

$$\exp\{\alpha\sigma\} = \cosh(\alpha) + \sigma \sinh(\alpha)\,, \quad \alpha \in \mathbb{R}.$$

Problem 2: The one-dimensional Ising model

Now we assume to have a one-dimensional lattice consisting of $(N+1)$ sites enumerated by i, i.e. $i \in \{1, 2, \ldots, N+1\}$. To each site is attached a "spin" degree of freedom (Ising model) $\sigma_i \in Z_2$. The total energy of this system shall be given by

$$H := -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}\,, \quad J \in \mathbb{R}.$$

Calculate the partition function (by using the results of Problem 1)

$$Z(\beta) := \sum_{\sigma_1 \in Z_2} \cdots \sum_{\sigma_{N+1} \in Z_2} \exp\{-\beta H\}\;, \quad \beta = 1/k_B T\,.$$

Compare this with the result for periodic boundary conditions, i.e.

$$Z_{PB}(\beta) := \sum_{\sigma_1 \in Z_2} \cdots \sum_{\sigma_N \in Z_2} \exp\{-\beta H\} \quad \text{with} \quad \sigma_{N+1} = \sigma_1\,.$$