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ASTRONOMY AND ASTROPHYSICS

How to monitor optimum exposure times for high resolution imaging modes?

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Abstract. Optimum exposure times for high resolution imaging modes are functions of the distribution of wind and turbulence in the Earth's atmosphere.

The theory which is developed here shows that the velocity of the tilt for the angle of arrival of light at the ground level, or equivalently the observed image motion velocity, is also statistically related to turbulent atmosphere motions.

The method of differential image motion which is actually used for monitoring the "Fried parameter", can be extended to one new differential image motion velocity method. It can therefore be used as a diagnostic to estimate exposure times needed to freeze the drastic effect of the wavefront corrugation evolution in high angular resolution imagery. Technically, this requires a growth of the sampling rate of ESO Differential Image Motion Monitor (DIMM).

Key words: turbulence – seeing – site testing – observational methods – interferometry

1. Introduction

The effective performances of the observing astronomical modes requiring high angular resolution depend primarily on the velocity of the wavefront corrugation evolution produced by atmospheric turbulence.

For example, the response time of an adaptive optic device, τ_{AO}, should be shorter than a typical coherence time, in order to "follow" the evolution of the distorted wavefront. In this case, the optimum response time can be calculated as a function of the motion of the diffusing atmospheric layers (Roddier et al. 1982), using the relation

$$\tau_{AO} = 0.31 \frac{r_0}{v^*}$$

where r_0 is the Fried parameter, and v^* the mean velocity for the turbulent layers displacement. This equation also holds for Michelson interferometry.

In speckle interferometry, the speckle life time or speckle boiling time, τ_{SP}, can be derived from a similar relation

$$\tau_{SP} = k \frac{r_0}{\Delta v}$$

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where Δv is the mean velocity dispersion for the turbulent layers, and k is a constant slightly depending on the formalism used (Aime et al. 1986).

The optimum exposure time to "freeze" the interference fringes in long baseline interferometers (considered as Michelson Interferometers), should depend on the direction of motion of the atmosphere with respect to the baseline (Colavita et al. 1987).

This paper presents a new method of image motion analysis. The method is based on the observed image motion velocity, which allows to derive physical quantities characterizing the wavefront evolution: i.e. $\overline{\Delta v}$, v^* and the direction of v^*.

The DIMM principle, useful for the measure of r_0 (Sarazin & Roddier 1990, Fried 1975), appears to be powerful for monitoring preceding quantites.

In Sect. 2, the variance for the motion velocity of a stellar image is calculated in the case of a simple atmosphere model. Section 3 describes the temporal behavior of the image motion velocity phenomenon.

As structure function is well adapted to the DIMM measurement, a structure function for the image motion velocity related to the preceding variance is developed in Sect. 4.

Then, Sect. 5 gives a more realistic motion velocity structure function derived from a complete model for the Earth's atmosphere.

The physics which can be made from these results, leading to the diagnostic of optimum exposure times is discussed in Sect. 6.

2. Variance for the motion velocity of a stellar image observed through a small telescope

The simple Taylor hypothesis is used to calculate, at the focus of a telescope with a diameter D, the variance for the motion velocity of one stellar image. Where D has to be smaller then r_0.

The near field approximation is also used in this model: only the phase variations of the wavefront at the ground level are considered.

Let us take $\alpha(x, y)$, the angle of arrival of light in the x direction and at the horizontal coordinates (x, y)

$$\alpha(x, y) = -\frac{\lambda}{2\pi} \frac{\partial}{\partial x} \varphi_0(x, y) \tag{2.1}$$

where $\varphi_0(x, y)$ represents the phase on the wavefront at the ground level. λ is the wavelength.

The Taylor hypothesis supposes that the atmosphere is constituted by a single and "frozen" diffusing layer, transported at the wind speed $\boldsymbol{v}$.

The frozen wavefront corrugation also should follow the layer in its horizontal translation. Then, we can describe the tilt velocity for the angle of arrival of light by

$$\frac{\partial\alpha}{\partial t}(x,y)=\frac{\partial x}{\partial t}\frac{\partial\alpha}{\partial x}(x,y)+\frac{\partial y}{\partial t}\frac{\partial\alpha}{\partial y}(x,y) \tag{2.2}$$

where

$$\begin{cases}\dfrac{\partial x}{\partial t}=|\boldsymbol{v}|\cos\theta\\ \dfrac{\partial y}{\partial t}=|\boldsymbol{v}|\sin\theta\end{cases} \tag{2.3}$$

θ represents the angle made by the direction of displacement of the wavefront with the $\boldsymbol{x}$ direction.

From (2.1, 2, 3), the covariance for the tilt velocity of the angle of arrival of light is:

$$B_{\partial\alpha/\partial t}(\mu,\eta)=\left\langle\frac{\partial\alpha}{\partial t}(x,y)\frac{\partial\alpha}{\partial t}(x+\mu,y+\eta)\right\rangle \tag{2.4}$$

$$\begin{aligned}B_{\partial\alpha/\partial t}(\mu,\eta)=\left(\frac{v\lambda}{2\pi}\right)^2\Bigg[&\cos^2\theta\left\langle\frac{\partial^2}{\partial x^2}\varphi_0(x,y)\frac{\partial^2}{\partial x^2}\varphi_0(x+\mu,y+\eta)\right\rangle\\&+\sin^2\theta\left\langle\frac{\partial^2}{\partial x\partial y}\varphi_0(x,y)\frac{\partial^2}{\partial x\partial y}\varphi_0(x+\mu,y+\eta)\right\rangle\\&+2\sin\theta\cos\theta\left\langle\frac{\partial^2}{\partial x^2}\varphi_0(x,y)\frac{\partial^2}{\partial x\partial y}\varphi_0(x+\mu,y+\eta)\right\rangle\Bigg]\,.\end{aligned} \tag{2.5}$$

As the process considered above is stationary and ergodic, its covariance is also equal to its autocorrelation function. From the Wiener-Khintchine theorem, the Fourier transform of the autocorrelation $B_{\partial\alpha/\partial t}(\mu,\eta)$ is the power spectrum density for tilt angle velocity of α : $W_{\partial\alpha/\partial t}(f_x,f_y)$, where $\boldsymbol{f}=(f_x,f_y)$ is a spatial frequency vector expressed in m^{-1}.

Relation (2.5) leads to:

$$\begin{aligned}W_{\partial\alpha/\partial t}(f_x,f_y)=&4\pi^2\lambda^2v^2(\cos^2\theta f_x^4+\sin^2\theta f_x^2f_y^2\\&+2\sin\theta\cos\theta f_x^3f_y)W_{\varphi_0}(f_x,f_y)\,.\end{aligned} \tag{2.6}$$

The power spectrum density $W_{\varphi_0}(f_x,f_y)$ for the phase φ_0 is given by F. Roddier (Roddier 1981) in the near field approximation (Appendix A)

$$W_{\varphi_0}(f_x,f_y)=0.229\,r_0^{-5/3}f^{-11/3}\,. \tag{2.7}$$

Considering now the pupil filter function

$$W_{\partial\alpha/\partial t,\mathrm{filt.}}(f_x,f_y)=W_{\partial\alpha/\partial t}(f_x,f_y)\cdot\left[\frac{2J_1(\pi Df)}{\pi Df}\right]^2 \tag{2.8}$$

where $W_{\partial\alpha/\partial t,\mathrm{filt.}}$ becomes the power spectrum density for image motion velocity filtered by the pupil function. J_1 is the first order Bessel function.

The variance for the image motion velocity of one stellar image seen through a telescope of diameter D is then

$$\begin{aligned}B_{\partial\alpha/\partial t}(0,0)&=\left\langle\left(\frac{\partial\alpha}{\partial t}(x,y)\right)^2\right\rangle\\&=\iint_{-\infty}^{+\infty}W_{\partial\alpha/\partial t,\mathrm{filt.}}(f_x,f_y)\mathrm{d}f\end{aligned} \tag{2.9}$$

We find (Appendix B) after integration of (2.9) and using (2.7, 6, 9),

$$B_{\partial\alpha/\partial t}(0,0)=0.128\lambda^2v^2r_0^{-5/3}D^{-7/3}(2\cos^2\theta+1)\,. \tag{2.10}$$

This variance is wavelength independent as $r_0^{-5/3}$ is proportional to λ^{-2}.

3. Temporal behavior of the stellar image motion velocity

The power spectrum density (2.8) can be expressed as a function of temporal frequencies: ν in Hz, by the use of the relation (Hufnagel 1978),

$$W_{\partial\alpha/\partial t}(f_x,f_y)=\frac{2}{v}\int_{-\infty}^{+\infty}W_{\partial\alpha/\partial t,\mathrm{filt.}}\left(\frac{\nu}{v}f_y\right)\mathrm{d}f_y\,. \tag{3.1}$$

Here v is supposed to be parallel to the $\boldsymbol{x}$ direction.

The following curves (Fig. 1) are obtained after numerical integration of (3.1). The parameters for the computation are:

(1) $D=0.07$ m.
(2) $r_0=0.1$ m for the visible.
(3) v has been chosen respectively equal to $10\,\mathrm{m\,s^{-1}}$, $20\,\mathrm{m\,s^{-1}}$, $30\,\mathrm{m\,s^{-1}}$.

Figure 1 shows that spectrum grows and shifts forward high frequencies when v increases.

4. Structure function for the motion velocity

The DIMM principle has shown its ability for the measure of seeing angle (Pedersen et al. 1988). By the help of differential method for the measure of image motion, vibrations or tracking error of telescope are naturally substracted to the signal of our interest.

The structure function for the motion velocity of stellar image $D_{\partial\alpha/\partial t}(\mu,\eta)$ is given by the relation

$$D_{\partial\alpha/\partial t}(\mu,\eta)=\left\langle\left|\frac{\partial\alpha}{\partial t}(x,y)-\frac{\partial\alpha}{\partial t}(x+\mu,y+\eta)\right|^2\right\rangle. \tag{4.1}$$

By definition, structure function is related to covariance,

$$D_{\partial\alpha/\partial t}(\mu,\eta)=2\left(B_{\partial\alpha/\partial t}(0,0)-B_{\partial\alpha/\partial t}(\mu,\eta)\right) \tag{4.2}$$

where

$$B_{\partial\alpha/\partial t}(\mu,\eta)=\iint_{-\infty}^{+\infty}W_{\partial\alpha/\partial t,\mathrm{filt.}}(f_x,f_y)\cos(2\pi\boldsymbol{\mu}\cdot\boldsymbol{f})\mathrm{d}f\,. \tag{4.3}$$

Unfortunately, this integral does not have a simple analytical solution, and, generally, one has to compute it numerically.

$B_{\partial\alpha/\partial t}(\mu,\eta)$ can be decomposed in one covariance for the parallel motion to the two subpupils: $B_{\partial\alpha\parallel/\partial t}(\mu,\eta)$, and one covariance for the perpendicular motion: $B_{\partial\alpha\perp/\partial t}(\mu,\eta)$.

It follows (Appendix C):

$$B_{\partial\alpha\parallel/\partial t}(\mu,0)=\lambda^2v^2r_0^{-5/3}[A(\mu,0)\cos^2\theta+B(\mu,0)\sin^2\theta] \tag{4.4}$$

$$B_{\partial\alpha\perp/\partial t}(\mu,0)=\lambda^2v^2r_0^{-5/3}[C(\mu,0)\cos^2\theta+B(\mu,0)\sin^2\theta]\,. \tag{4.5}$$

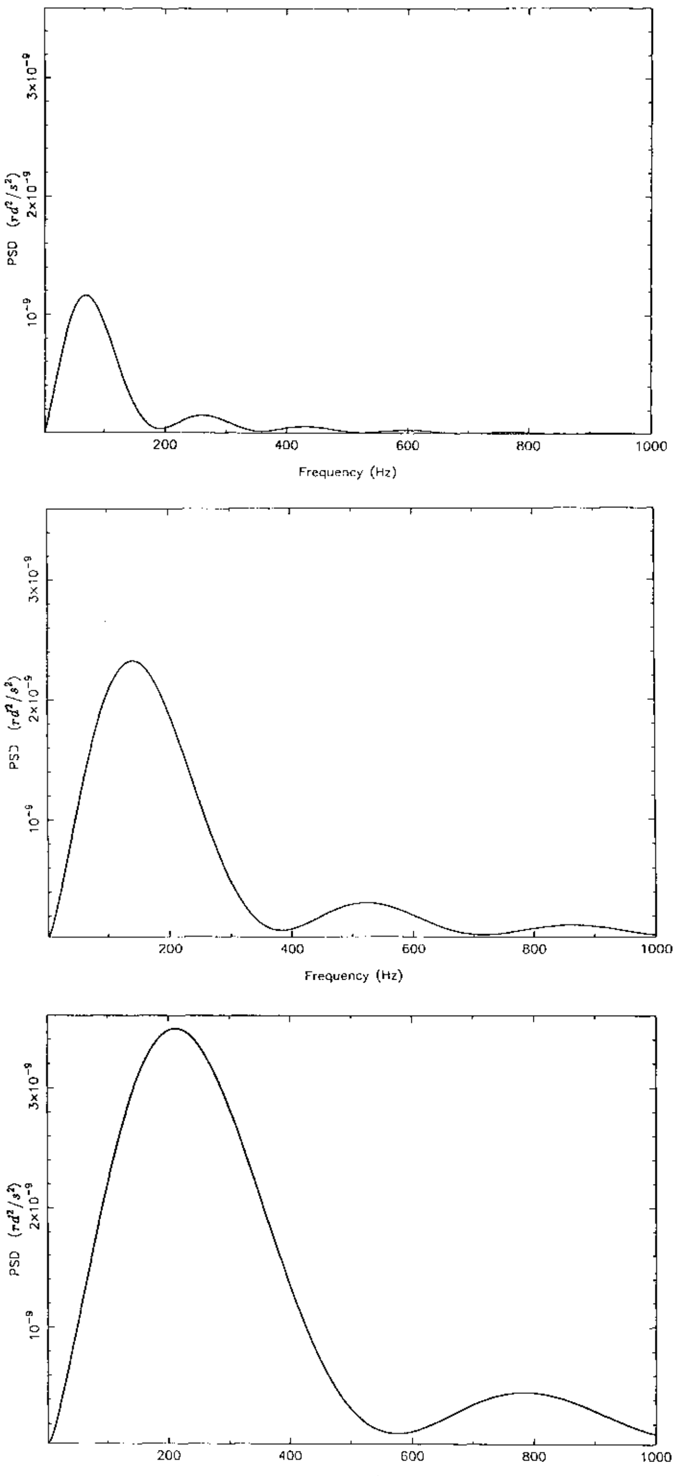


Fig. 1. Power spectrum density for stellar image motion velocity drawn for several wind speeds: $v = 10\,\mathrm{m\,s^{-1}}$, $20\,\mathrm{m\,s^{-1}}$, and $30\,\mathrm{m\,s^{-1}}$

The coefficients $A(\mu,0)$, $B(\mu,0)$, $C(\mu,0)$ are presented in Table 1 for a pupil diameter $D = 0.07\,\mathrm{m}$, and for several pupil separations μ.

Choosing for example a pupil separation of 0.20 m, structure functions for $\|$ and $\perp$ image motion velocites are then (2.10), (4.2, 4, 5),

Table 1. Covariance constants for several pupil separations and for a pupil diameter equal to 0.07 m

Separation: μ (m)	$A(\mu,0)$	$B(\mu,0)$	$C(\mu,0)$
0.15	−3.58	−4.02	9.20
0.20	−1.69	−2.15	4.57
0.25	−0.98	−1.30	2.55
0.30	−0.62	−0.86	1.75

$$D_{\partial\alpha\|/\partial t} = \lambda^2 r_0^{-5/3} v^2 (128.8 + 252.8 \cos^2\theta) \quad (4.6)$$

$$D_{\partial\alpha\perp/\partial t} = \lambda^2 r_0^{-5/3} v^2 (373.4 - 245.3 \cos^2\theta)\,, \quad (4.7)$$

where covariances $B_{\partial\alpha\|/\partial t}(\mu,0)$ and $B_{\partial\alpha\perp/\partial t}(\mu,0)$ represent only a contribution smaller than 3% in the above equations (4.6) and (4.7).

5. Atmosphere constituted by multiple thin turbulent layers

To adapt this structure function to the real atmosphere, one can consider that turbulence is located in several and independent thin diffusing layers.

Holding the near field approximation, each of these layers adds its effect to the deviation of the angle of arrival α:

$$\frac{\partial\alpha}{\partial t}(x,y) = \sum_{h=0}^{H} \frac{\partial\alpha_h}{\partial t}(x,y)\,, \quad (5.1)$$

where α_h is the deviation caused by the diffusing layer located at altitude h. α is the total deviation observed at the ground level.

From Appendix D, it follows:

$$B_{\partial\alpha/\partial t}(\mu,\eta) = \left(\frac{\lambda}{2\pi}\right)^2 \left[\left(\overline{v_x}^2 + \frac{\overline{\Delta v}^2}{2}\right)\left\langle \frac{\partial\alpha}{\partial x}(x,y)\frac{\partial\alpha}{\partial x}(x+\mu,y+\eta)\right\rangle + \left(\overline{v_y}^2 + \frac{\overline{\Delta v}^2}{2}\right)\left\langle \frac{\partial\alpha}{\partial y}(x,y)\frac{\partial\alpha}{\partial y}(x+\mu,y+\eta)\right\rangle\right] \quad (5.2)$$

Therefore, this statement leads easily to a new variance for the image motion velocity and a new structure function in the case of a more complete atmospheric pattern.

From (2.10):

$$B_{\partial\alpha/\partial t}(0,0) = 0.128\lambda^2 r_0^{-5/3} D^{-7/3} (\bar{v}^2(2\cos^2\bar{\theta} + 1) + 2\bar{\Delta} v^2) \quad (5.3)$$

and from (4.6, 7):

$$D_{\partial\alpha\|/\partial t} = \lambda^2 r_0^{-5/3} (\bar{v}^2(128.8 + 252.8\cos^2\bar{\theta}) + 255.2\overline{\Delta v}^2) \quad (5.4)$$

$$D_{\partial\alpha\perp/\partial t} = \lambda^2 r_0^{-5/3} (\bar{v}^2(373.4 - 245.3\cos^2\bar{\theta}) + 250.6\overline{\Delta v}^2) \quad (5.5)$$

where $\cos^2\theta$ has been averaged for the terms containing $\overline{\Delta v}^2$ which have isotropic effect on the motion velocity of a stellar image.

$\bar{v}$ is an averaged velocity for the turbulent layers, $\bar{\theta}$ its direction,

$$\bar{v}^2 = \left[\frac{\int_0^H v(h) C_n^2(h)\mathrm{d}h}{\int_0^H C_n^2(h)\mathrm{d}h}\right]^2 \quad (5.6)$$

and $\overline{\Delta v}$ is the standard deviation of the distribution of wind velocity weighted by the turbulent structure coefficient,

$$\overline{\Delta v}^2 = \left[\frac{\int_0^H (\Delta v(h))^2 C_n^2(h)\mathrm{d}h}{\int_0^H C_n^2(h)\mathrm{d}h}\right] \tag{5.7}$$

6. Discussion

The image motion velocity structure functions (5.4) and (5.5) depend on four physical quantites of importance r_0, $\overline{\Delta v}$, $\bar{v}$ and $\bar{\theta}$, leading to the diagnostic of optimum exposure time for high resolution imaging modes.

(1) The Fried parameter r_0 can be supposed known by applying the DIMM theory.

(2) $\overline{\Delta v}$ is the same parameter which was described in the introduction. It has been measured by radiosonde wind profile ballons during LASSCA campaign (VLT report No. 60) and led to a calculated speckle life time value found with a good agreement with some direct measurements.

(3) $\bar{v}$ is not exactly the same averaged velocity for the turbulent atmosphere v^* that we speak about in the introduction,

$$v^* = \left[\frac{\int_0^H |v(h)|^{5/3} C_n^2(h)\mathrm{d}h}{\int_0^H C_n^2(h)\mathrm{d}h}\right]^{3/5} . \tag{6.1}$$

Nevertheless it can be a good estimator of v^*.

(4) $\bar{\theta}$ is another parameter of interest, because it should help for modelizing the non-isotropic effect for fringe exposure time of long baseline interferometry.

It should also relate more accurately the optimum response time for adaptive optic to the atmosphere.

Let us now give a physical meaning to the method and show how easily the considered quantities can be measured with a DIMM:

Normally structure functions (5.4) and (5.5) must be used in their complete expressions, but as covariances $B_{\partial\alpha\parallel/\partial t}$ and $B_{\partial\alpha\perp/\partial t}$ are much smaller than $B_{\partial\alpha/\partial t}(0,0)$ in Eqs. (4.6) and (4.7), their effects will be neglected in this discussion for the sake of simplicity. The structure function for image motion velocity can hence be approximated by twice time the variance $B_{\partial\alpha/\partial t}(0,0)$ (5.3)

$$D_{\partial\alpha/\partial t} = 0.256\lambda^2 r_0^{-5/3} D^{-7/3}(\bar{v}^2(2\cos^2\bar{\theta}+1)+2\overline{\Delta v}^2) . \tag{6.2}$$

Table 2 gives some didactic particular cases showing the shape of the variance $B_{\partial\alpha/\partial t}(0,0)$ in any direction of measure in the horizontal plane.

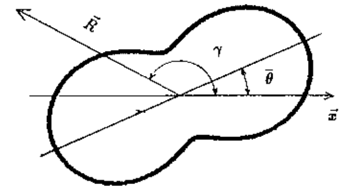
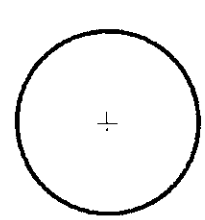
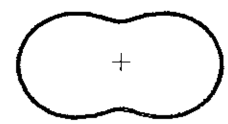
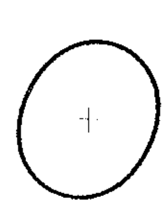
(1) When $\overline{\Delta v}^2 = 0$, then the graphic representation of the variance is a "rosette des vents" elongated in the direction of wavefront corrugation displacement $\bar{\theta}$. Long axis of the rosette-small axis ratio is then equal to 3. This particular case should exist if Taylor hypothesis is satisfied.

(2) $\bar{v}^2 = 0$, then the value of the variance is not depending on the direction of observation. A circle proportional to $\overline{\Delta v}^2$ for the plane representation of the variance reveals the isotropic phasis boiling phenomenon on the wavefront.

(3) More realistic conditions are probably a mixing of $\overline{\Delta v}$ and $\bar{v}$ in any proportions, leading to a long axis–small axis ratio for the rosette between 1 and 3.

Clearly, $\bar{\theta}$ is still given by the direction of elongation, and,

Table 2. Some representations of agitation velocity variance in any direction of measure for particular atmospheric parameters. $\bar{\theta}$ = Angle between the x axis and the wavefront corrugation displacement direction. γ = Angle between the x axis and the direction of measure for the variance. R polar radius, $|R| = 3\,10^{-7}\,\mathrm{rd}^2/\mathrm{s}^2$

Atmospheric parameters	Effect on the wavefront	Image motion velocity variance
$\overline{\Delta v} = 0$ $\bar{v} = 10\,\mathrm{m\,s^{-1}}$ $\theta = 20°$	Taylor hypothesis	
$\overline{\Delta v} = 10\,\mathrm{m\,s^{-1}}$ $\bar{v} = 0$	Boiling phase	
$\overline{\Delta v} = 4\,\mathrm{m\,s^{-1}}$ $\bar{v} = 8\,\mathrm{m\,s^{-1}}$ $\theta = 0°$	Boiling + Taylor	
$\overline{\Delta v} = 8\,\mathrm{m\,s^{-1}}$ $\bar{v} = 4\,\mathrm{m\,s^{-1}}$ $\bar{\theta} = 70°$		

from the measurement of $B_{\partial\alpha/\partial t}$ for $\theta = 0°$ and for $\theta = 90°$, $\bar{v}$ and $\overline{\Delta v}$ can be recovered using (6.2)

$$\bar{v} = \frac{D_{(\partial\alpha/\partial t),\theta=0} - D_{(\partial\alpha/\partial t),\theta=90}}{0.512\lambda^2 r_0^{-5/3} D^{-7/3}} \tag{6.3}$$

$$\overline{\Delta v} = \frac{D_{(\partial\alpha/\partial t),\theta=0} - 3D_{(\partial\alpha/\partial t),\theta=90}}{1.024\lambda^2 r_0^{-5/3} D^{-7/3}} . \tag{6.4}$$

The method presented here will be tested at ESO La Silla, using an adapted version of the DIMM aiming at measuring image motion velocity. This instrument should resolve the evolution of motion on the wavefront and therefore sample positions of a stellar image at a frequency around 500 Hz, as shown in Sect. 3.

The future monitoring of these physical parameters, and therefore, the monitoring of the optimum exposure times will aim to adapt the next astronomical program in high resolution imaging modes to the random behavior of the turbulent atmosphere.

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Appendix A

$W_{\varphi_0}(f_x, f_y)$ given by Roddier (Roddier 1981) in the near field approximation is

$$W_{\varphi_0}(f_x, f_y) = 0.38\lambda^{-2} f^{-11/3} \int_0^H C_n^2(h)\mathrm{d}h$$

valid in the inertial range of Kolmogorov

$$\frac{1}{L_0} < f < \frac{1}{l_0}$$

where L_0 is the outer scale of turbulence and l_0 the inner scale.

$W_{\varphi_0}(f_x, f_y)$ can be related to the Fried parameter,

$$r_0 = \left[0.423\, k^2 \int_0^H C_n^2(h)\mathrm{d}h\right]^{3/5}$$

where k is the wavenumber of the vibration,

$$k = \frac{2\pi}{\lambda}$$

leading to

$$W_{\varphi_0}(f_x, f_y) = 0.0229 r_0^{-5/3} f^{-11/3} .$$

Appendix B

Relation (2.11) gives

$$B_{\partial\alpha/\partial t}(0,0) = 4\pi^2\lambda^2 v^2 \iint_{-\infty}^{+\infty} \left(\cos^2\theta f_x^4 + \sin^2\theta f_x^2 f_y^2 + 2\sin\theta\cos\theta f_x^2 f_y^2\right)\left(0.0229 r_0^{-5/3} f^{-11/3}\right) \times \left[\frac{2J_1(\pi \mathrm{d} f)}{\pi D f}\right]^2 \mathrm{d}f .$$

An approximation is made when extending the Kolmogorov field of validity for low frequencies between 0 and $1/L_0$. For high frequencies side, formalism is correct, the pupil filter cut off frequency being smaller than $1/l_0$.

By passing to a polar coordinate system:

$$\begin{cases} f_x = \rho\cos\gamma \\ f_y = \rho\sin\gamma \end{cases}$$

and after integration over γ, the integral becomes proportional to

$$B_{\partial\alpha/\partial t}(0,0) \sim \int_0^{+\infty} \rho^{-2/3} J_1^2(\pi D\rho)\mathrm{d}\rho$$

which has an analytical solution (Gradshteyn & Ryzhik 1980).

Since,

$$B_{\partial\alpha/\partial t}(0,0) = 0.128\lambda^2 v^2 r_0^{-5/3} D^{-7/3}(2\cos^2\theta + 1) .$$

Appendix C

According to

$$B_{\partial\alpha\|/\partial t}(\mu, 0) = \iint_{-\infty}^{+\infty} W_{\partial\alpha/\partial t,\mathrm{filt.}}(f_x, f_y)\cos(2\pi\mu f_x)\mathrm{d}f .$$

Substituting θ by $\theta + \pi/2$ and permuting f_x and f_y in $W_{\partial\alpha/\partial t,\mathrm{filt.}}(f_x, f_y)$ leads to,

$$B_{\partial\alpha\perp/\partial t}(\mu, 0) = \int_{-\infty}^{+\infty} W_{\partial\alpha/\partial t,\mathrm{filt.},\theta\to\theta+\pi/2}(f_y, f_x)\cos(2\pi\mu f_x)\mathrm{d}f .$$

Then both covariances can be written

$$B_{\partial\alpha\|/\partial t}(\mu, 0) = \lambda^2 v^2 r_0^{-5/3}(A\cos^2 t + B\sin^2\theta)$$

$$B_{\partial\alpha\perp/\partial t}(\mu, 0) = \lambda^2 v^2 r_0^{-5/3}(B\cos^2\theta + C\sin^2\theta)$$

where

$$A = \frac{16}{D^2}\iint_{\infty}^{+\infty} f_x^4 f^{-17/3} J_1^2(\pi D f)\cos(2\pi\mu f_x)\mathrm{d}f$$

$$B = \frac{16}{D^2}\iint_{-\infty}^{+\infty} f_x^2 f_y^2 f^{-17/3} J_1^2(\pi D f)\cos(2\pi\mu f_x)\mathrm{d}f$$

$$C = \frac{16}{D^2}\iint_{-\infty}^{+\infty} f_y^4 f^{-17/3} J_1^2(\pi D f)\cos(2\pi\mu f_x)\mathrm{d}f .$$

Appendix D

$$B_{\partial\alpha/\partial t}(\mu, \eta) = \left\langle \sum_{h=0}^{H} \frac{\partial\alpha_h}{\partial t}(x, y) \sum_{h=0}^{H} \frac{\partial\alpha_h}{\partial t}(x+\mu, y+\eta) \right\rangle . \qquad \text{(D1)}$$

Covariance becomes

$$B_{\partial\alpha/\partial t}(\mu, \eta) = \left\langle \sum_{h=0}^{H} \frac{\partial\alpha_h}{\partial t}(x, y) \frac{\partial\alpha_h}{\partial t}(x+\mu, y+\eta) \right\rangle + 2\left\langle \sum_{h=0}^{H}\sum_{j=h}^{H} \frac{\partial\alpha_h}{\partial t}(x, y) \frac{\partial\alpha_j}{\partial t}(x+\mu, y+\eta) \right\rangle . \qquad \text{(D2)}$$

Since the diffusing layers are assumed to be independent, the intercorrelation component of this relation is equal to zero. Let us consider that the layers are so thin, that we can write continuously over the atmosphere:

$$B_{\partial\alpha/\partial t}(\mu, \eta) = \left\langle \int_0^H \frac{\partial\alpha_h}{\partial t}(x, y) \frac{\partial\alpha_h}{\partial t}(x+\mu, y+\eta)\mathrm{d}h \right\rangle \qquad \text{(D3)}$$

it comes,

$$B_{\partial\alpha/\partial t}(\mu,\eta) = \left(\frac{\lambda}{2\pi}\right)^2 \int_0^H \left(v_h^2 \cos^2\theta_h \left\langle \frac{\partial\alpha_h}{\partial x}(x,y)\frac{\partial\alpha_h}{\partial x}(x+\mu,y+\eta)\right\rangle \right.$$
$$+ v_h^2 \sin^2\theta_h \left\langle \frac{\partial\alpha_h}{\partial y}(x,y)\frac{\partial\alpha_h}{\partial y}(x+\mu,y+\eta)\right\rangle$$
$$\left. +2v_h^2 \sin\theta_h \cos\theta_h \left\langle \frac{\partial\alpha_h}{\partial x}(x,y)\frac{\partial\alpha_h}{\partial y}(x+\mu,y+\eta)\right\rangle \right) \mathrm{d}h\,. \quad \text{(D4)}$$

The term in $\sin\theta\cos\theta$ dos not have contribution for the calculation of parallel and perpendicular covariance.

It remains,

$$B_{\partial\alpha/\partial t}(\mu,\eta) = \left(\frac{\lambda}{2\pi}\right)^2 \int_0^H \left(v_h^2 \cos^2\theta_h \left\langle \frac{\partial\alpha_h}{\partial x}(x,y)\frac{\partial\alpha_h}{\partial x}(x+\mu,y+\eta)\right\rangle \right.$$
$$\left. +v_h^2 \sin^2\theta_h \left\langle \frac{\partial\alpha_h}{\partial y}(x,y)\frac{\partial\alpha_h}{\partial y}(x+\mu,y+\eta)\right\rangle \right) \mathrm{d}h\,. \quad \text{(D5)}$$

The refractive index structure coefficient $C_n^2(h)$ expresses the amount of atmospheric turbulence for optical propagation located on the altitude layer h, using it,

$$\left\langle \frac{\partial\alpha_h}{\partial x}(x,y)\frac{\partial\alpha_h}{\partial x}(x,y)\right\rangle$$
$$= \frac{C_n^2(h)}{\int_0^H C_n^2(h)\mathrm{d}h} \left\langle \frac{\partial\alpha}{\partial x}(x,y)\frac{\partial\alpha}{\partial x}(x+\mu,y+\eta)\right\rangle . \quad \text{(D6)}$$

By writing

$$\begin{cases} \boldsymbol{v}_h \cos\theta_h = \overline{\boldsymbol{v}_x} + \Delta\boldsymbol{v}_x(h) \\ \boldsymbol{v}_h \sin\theta_h = \overline{\boldsymbol{v}_y} + \Delta\boldsymbol{v}_y(h) \end{cases} \quad \text{(D9–D10)}$$

with

$$\begin{cases} \overline{\boldsymbol{v}_x} = \dfrac{\int_0^H C_n^2(h)\boldsymbol{v}_h \cos\theta_h \mathrm{d}h}{\int_0^H C_n^2(h)\mathrm{d}h} \\ \overline{\boldsymbol{v}_y} = \dfrac{\int_0^H C_n^2(h)\boldsymbol{v}_h \sin\theta_h \mathrm{d}h}{\int_0^H C_n^2(h)\mathrm{d}h} \end{cases} \quad \text{(D11–D12)}$$

$\overline{\boldsymbol{v}_x}$ and $\overline{\boldsymbol{v}_y}$ are averaged velocity for the layers weighted by their degree of turbulence $C_n^2(h)$.

$\Delta\boldsymbol{v}_x(h)$ and $\Delta\boldsymbol{v}_y(h)$ are dispersion of velocity for the wind speed and for the located layer h.

Equation (D5) becomes

$$B_{\partial\alpha/\partial t}(\mu,\eta) = \left(\frac{\lambda}{2\pi}\right)^2 \cdot \frac{1}{\int_0^H C_n^2(h)\mathrm{d}h} \left[\left(\int_0^H \overline{v_x}^2 C_n^2(h)\mathrm{d} \right.\right.$$
$$\left. + \int_0^H C_n^2(h)\Delta v_x^2(h)\mathrm{d}h + 2\int_0^H \overline{\boldsymbol{v}_x}\cdot\Delta\boldsymbol{v}_x(h)C_x^2(h)\mathrm{d}h \right)$$
$$\times \left\langle \frac{\partial\alpha}{\partial x}(x,y)\frac{\partial\alpha}{\partial x}(x+\mu,y+\eta)\right\rangle$$
$$+ \left(\int_0^H \overline{v_y}^2 C_n^2(h)\mathrm{d}h + \int_0^H Cn^2(h)\Delta v_y^2(h)\mathrm{d}h \right.$$
$$\left. +2\int_0^H \overline{\boldsymbol{v}_y}\Delta\boldsymbol{v}_y(h)C_x^2(h)\mathrm{d}h \right)$$
$$\left. \times \left\langle \frac{\partial\alpha}{\partial y}(x,y)\frac{\partial\alpha}{\partial y}(x+\mu,y+\eta)\right\rangle \right] . \quad \text{(D13)}$$

By definition of $\overline{\boldsymbol{v}_x}$ and $\overline{\boldsymbol{v}_y}$,

$$\begin{cases} \int_0^H \overline{\boldsymbol{v}_x}\Delta\boldsymbol{v}_x(h)C_n^2(h)\mathrm{d}h = 0 \\ \int_0^H \overline{\boldsymbol{v}_y}\Delta\boldsymbol{v}_y(h)C^2n(h)\mathrm{d}h = 0 \end{cases} \quad \text{(D14–D15)}$$

then,

$$B_{\partial\alpha/\partial t}(\mu,\eta) = \left(\frac{\lambda}{2\pi}\right)^2 \left[\left(\overline{v_x}^2 + \overline{\Delta v_x}^2\right) \left\langle \frac{\partial\alpha}{\partial x}(x,y)\frac{\partial\alpha}{\partial x}(x+\mu,y+\eta)\right\rangle \right.$$
$$\left. + \left(\overline{v_y}^2 + \overline{\Delta v_y}^2\right) \left\langle \frac{\partial\alpha}{\partial y}(x,y)\frac{\partial\alpha}{\partial y}(x+\mu,y+\eta)\right\rangle \right] \quad \text{(D16)}$$

with

$$\begin{cases} \overline{\Delta v_x}^2 = \dfrac{\int_0^H \Delta v_x^2(h)C_n^2(h)\mathrm{d}h}{\int_0^H (C_n^2(h)\mathrm{d}h} \\ \overline{\Delta v_y}^2 = \dfrac{\int_0^H \Delta v_y^2(h)C_n^2(h)\mathrm{d}h}{\int_0^H C_n^2(h)\mathrm{d}h} \end{cases} \quad \text{(D17–D18)}$$

Physically, the effect of $\overline{\Delta v_x}^2$ and $\overline{\Delta v_y}^2$ is to introduce on the wavefront a boiling component for the phasis of the incoming light. Though the velocity dispersion has a main direction, the statistic of this phasis boiling on the wavefront is isotropic.

For this reason, one can simplify the relation (5.15) by saying:

$$\overline{\Delta v_x}^2 = \overline{\Delta v_y}^2 = \frac{\overline{\Delta v^2}}{2} \quad \text{(D19)}$$

with

$$\overline{\Delta v}^2 = \frac{\int_0^H \left(\overline{\Delta v_x}^2(h) + \overline{\Delta v_y}^2(h)\right) C_n^2(h)\mathrm{d}h}{\int_0^H C_n^2(h)\mathrm{d}h} \quad \text{(D20)}$$

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