# THE TEMPERATURE STRUCTURE FUNCTION FOR COMPLEX TERRAIN

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## ABSTRACT

Design studies are being conducted with the aim of constructing a very large telescope (ø 16 m) to observe the night-time sky. A numerical model is developed to calculate the temperature structure function on top of the mountain and the results are compared to measurements. The model consists of an equation for the wind-components, where the turbulence model is the well-known $\varepsilon - E$ model, an equation for the temperature variance and its dissipation. From the values for $\varepsilon$ and $\varepsilon_\theta$ we derive the temperature structure function

$$C_t^2 = 1.6 \cdot \varepsilon_\theta \cdot \varepsilon^{-1/3}, \qquad (1)$$

(Wyngaard et al, 1971). The numerical results were compared to $C_t^2$ derived from the fine-scale temperature fluctuation measurements. The results show the impressive achievement of this model despite its severe approximations.

## INTRODUCTION

The European Southern Observatory (ESO) wants to construct a new Very Large Telescope (VLT) in Chile. They are looking for a mountain with optimal conditions for observation purposes. This means that the wind and turbulence fields have to meet certain criteria. For the contract the wind field was modeled over some chosen sites with WASP (Troen et al, 1989). Also some possible alterations in the site were considered like cutting the top of the mountain. The other part of the contract here described comprised modeling the 'seeing', which is a function of the temperature structure function $C_T^2$. The question concerned the possible effects of different mountains on the seeing.

## DESCRIPTION OF THE MODEL

The situation at the telescope site to be considered is nighttime flow over a steep hill, without separation. We confine ourselves to situations with higher winds so that the stability is weakly stable. In such situations the effect of the orography on the wind is much larger than the stability effects, and the buoyancy terms in the 'flow' equations (U, E, $\varepsilon$) can be neglected. This means a decoupling from the 'temperature' equations (T, $\overline{\theta^2}, \varepsilon_\theta$). As stationarity is required we do not solve the temperature equation, which is basically instationary in stable situations, but assume that the temperature profile is lifted over the hill.
In our two equation (E-$\varepsilon$) turbulence model the fluxes are approximated by a gradient approach with en eddy diffusivity $K_m = c_\mu \frac{E^2}{\varepsilon}$:

$$\begin{aligned}
\overline{u_i u_j} &= -K_m \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \\
-\frac{1}{2}\overline{u_j u_i^2} - \overline{u_i p} &= \frac{1}{2} K_E \frac{\partial \overline{u_i^2}}{\partial x_j} = K_E \frac{\partial E}{\partial x_j} \\
-\overline{u_j \varepsilon'} &= K_\varepsilon \frac{\partial \varepsilon}{\partial x_j} \\
\overline{u\theta} &= -K_h \frac{\partial \Theta}{\partial x} \\
\overline{w\theta} &\equiv u_* \Theta_* = -K_h \frac{\partial \Theta}{\partial z} \\
\overline{u_j \theta^2} &= -2K_h \frac{\partial \overline{\theta^2}}{\partial x_j}
\end{aligned} \qquad (2)$$

where $K_E = K_m$, $K_\varepsilon = K_m/\sigma_\varepsilon$ and $K_h = \alpha K_m$.
The constants $\sigma_\varepsilon$ and $\alpha^{-1}$ are called Prandtl numbers.

Substituting these flux modeling in the 'flow' equations for stationary flow without buoyancy effects we have for

momentum:

$$U_j \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left( K_m \frac{\partial U_i}{\partial x_j} \right) - \frac{1}{\rho_0}\frac{\partial P}{\partial x_i}. \qquad (3)$$

kinetic energy:

$$U_j \frac{\partial E}{\partial x_j} = \frac{\partial}{\partial x_j}\left( K_m \frac{\partial E}{\partial x_j} \right) + K_m \left( \frac{\partial U_i}{\partial x_j} \right)^2 - \varepsilon. \qquad (4)$$

temperature variance:

$$U_i \frac{\partial \overline{\theta^2}}{\partial x_i} = \frac{\partial}{\partial x_j}\left( K_h \frac{\partial \overline{\theta^2}}{\partial x_j} \right) + 2K_h \left( \frac{\partial \Theta}{\partial x_i} \right)^2 - 2\varepsilon_\theta. \qquad (5)$$

The modeling of the dissipation equation can be found in Hanjalic and Launder, 1972:

$$U_j \frac{\partial}{\partial x_j}\varepsilon = \frac{\partial}{\partial x_j}\left( K \frac{\partial \varepsilon}{\partial x_j} \right) + 2c_{\varepsilon 1} K_m \left( \frac{\partial U_i}{\partial x_j} \right)^2 \frac{\varepsilon}{q^2} - 2c_{\varepsilon 2}\frac{\varepsilon^2}{q^2}, \qquad (6)$$

The modeling of the thermal dissipation equation can be found in Newman et al, 1981:

$$\frac{\partial}{\partial x_j}(U_j \varepsilon_\theta) = \frac{\partial}{\partial x_j}\left( \frac{K_h}{\sigma_{\varepsilon_\theta}} \frac{\partial \varepsilon_\theta}{\partial x_j} \right) + a_1 K_h \left( \frac{\partial \Theta}{\partial x_i} \right)^2 \frac{\varepsilon_\theta}{\overline{\theta^2}} - a_2 \frac{\varepsilon_\theta^2}{\overline{\theta^2}} - a_3 \frac{\varepsilon \varepsilon_\theta}{q^2}, \qquad (7)$$

## CONSTANTS IN THE EQUATIONS

The constants in the 'flow' part of the model for the atmosphere (Duynkerke,1982) can be determined to be $c_\mu = A^{-2} = (5.48)^{-2} = 0.033$, $c_{\epsilon 2} = 1.83$, $c_{\epsilon 1} = 1.46$ and $\sigma_\epsilon = 2.38$ .

The neutral and stable surface layer profiles found on flat terrain (Businger,1971) satisfy the $\overline{\theta^2}$ equation. For the $\varepsilon_\theta$ equation the situation becomes too involved if stability is taken into account and in this model the coefficients are tuned to the neutral situation. It follows that

$$\frac{u_\star^2 \Theta_\star^2}{z^2}\left[\frac{1}{\sigma_\theta} + \frac{1}{\kappa^2}\left[a_1\frac{(0.74)^2}{B} - a_2\frac{(0.74)^2}{B} - a_3\frac{(0.74)}{2A}\right]\right] = 0. \quad (8)$$

We assumed that $\sigma_\theta = \sigma_\epsilon = 2.38$. There are various solutions to this equation. We tested $\{a_1, a_2, a_3\}=\{1,1,1\}$, $\{2.36, 2.02, 1.5\}$ and $\{5, 3.38, 3.38\}$ for the atmosphere, where the second combination keeps $a_2$ as given by Newman and the other two combinations are chosen such that the $a_i$ are wide apart. We found that the results for the temperature structure function $C_T^2$, (our ultimate goal) are the same on flat terrain no matter what the constants are.

The weakly stable profiles turned out to be close to a solution of the $\varepsilon_\theta$ equation with these sets of coefficients, in the sense that only slight changes in $C_T^2$ occurred downstream.

## TEMPERATURE STRUCTURE FUNCTION

The temperature structure function is defined as

$$C_T^2 = \frac{\overline{(\Theta(\vec{r_o}) - \Theta(\vec{r} + \vec{r_o}))^2}}{r^{\frac{2}{3}}} \quad (9)$$

and can be related to the 3D temperature spectra $S_T$ normalized as

$$\overline{\theta^2} = \int_0^\infty S_T(k) d(k).$$

Scaling the temperature spectrum with the parameters of the inertial subrange gives (Tennekes and Lumley, 1984 p. 283 with $S_T = 2E_\theta$)

$$S_T(k) = \beta\, \varepsilon_\theta\, \varepsilon^{-\frac{1}{3}} k^{-\frac{5}{3}}.$$

Here $\varepsilon_\theta$ is defined (as before) to be the rate of dissipation of half the temperature variance.

From the definition of $S_T$ and $C_T^2$ it follows

$$C_T^2 = \gamma\, \varepsilon_\theta\, \varepsilon^{-\frac{1}{3}}, \quad (10)$$

where Tennekes and Lumley's variables (1984) give $\gamma = 2.4$, Dutton and Panofsky (1983, p182) $\gamma = 2.8$ and Wyngaard et al.(1971) $\gamma = 3.2$. All authors agree that the constants are very difficult to measure and a factor 2 in uncertainty is not uncommon.

Wyngaard 1973, page 128 finds from measurements for the neutral and stable case

$$C_T^2 \frac{(\kappa z)^{\frac{2}{3}}}{\Theta_\star^2} = 2.66(1 + 2.4(\frac{z}{L})^{\frac{2}{3}})$$

which for neutral stability with $\kappa = .4$ reads $C_T^2 \frac{(\kappa z)^{\frac{2}{3}}}{\Theta_\star^2} = 2.66$ and which has as very stable limit $C_T^2 \frac{z^{\frac{2}{3}}}{\Theta_\star^2} = 11.76\xi^{\frac{2}{3}}$.

Using the surface layer profiles

$$\begin{aligned} \varepsilon &= \frac{u_\star^3}{\kappa z}\phi_m(\xi), \\ \varepsilon_\theta &= \frac{u_\star \Theta_\star^2}{\kappa z}\phi_h(\xi), \end{aligned} \quad (11)$$

where the stability parameter $\xi$ and Monin-Obukhov length $L$ are given by

$$\begin{aligned} \xi &= \frac{z}{L}, \\ L &= -\frac{\Theta_0}{\kappa g}\frac{u_\star^3}{\overline{w\theta_0}} = \frac{\Theta_0}{\kappa g}\frac{u_\star^2}{\Theta_\star}. \end{aligned} \quad (12)$$

in the scaling equation for $C_T^2$

$$C_T^2 = \gamma\frac{\varepsilon_\theta}{\varepsilon^{\frac{1}{3}}} = \gamma\frac{\Theta_\star^2}{(\kappa z)^{\frac{2}{3}}}\frac{\phi_h}{\phi_m^{\frac{1}{3}}} \quad (13)$$

For stable conditions we have the stability functions are $\phi_m = 1 + 4.7\xi$ and $\phi_h = .75 + 4.7\xi$. This gives the measured neutral limit mentioned above if $\gamma = 3.6$ and the very stable limit if $\gamma = 2.3$.

In our model we calculate with neutral $\phi_m$ and stable $\phi_h$, thus

$$\gamma = 2.66\left[1 + 2.4\xi^{\frac{2}{3}}\right][.75 + 4.7\xi]^{-1} \quad (14)$$

E.g. for $\xi = 1$ we have $\gamma = 1.6$. If necessary, a different value of $\gamma$ can directly be applied to the results by a simple scaling if the $C_T^2$ axis and all conclusions are still true in a relative sense.

## INNER LAYER THEORY

The relation between the wind profile at the top and the inlet profile in 2D cases can be described by inner layer theory (Jackson and Hunt, 1975).

For a hill with height $h$, width $L$ and roughness $z_0$

$$\frac{\ell}{L}\ln\left(\frac{\ell}{z_o}\right) = 2\kappa^2 \quad \text{and} \quad (15)$$

$$\Delta S_{max} \sim 2h/L, \quad (16)$$

$\ell$ is the inner layer height where the speedup $S$ is maximum.

With an inlet $U_0$ the differential speedup is defined as

$$\Delta S(x,z) \equiv \frac{U(x,z) - U_o(z)}{U_o(z)} = \frac{\Delta u_\star}{u_\star}. \quad (17)$$

Our model will show that the changes in $\varepsilon_\theta$ are much smaller than the ones in $\varepsilon$. If $\varepsilon$ at the top is taken equal to the inlet (frozen thermal turbulence) and inserting surface layer scaling for $\varepsilon$ we get

$$C_T^2 = 1.6\, \varepsilon_\theta\, \varepsilon^{-\frac{1}{3}} = 1.6\, \varepsilon_\theta\, \frac{\kappa z}{u_\star}. \qquad (18)$$

Applying inner layer theory we get with $x_o$ the x-coordinate of the inlet and $x_t$ the x-coordinate of the top

$$C_T^2(x_t, z = l) = \frac{1}{1 + \frac{\Delta u_\star}{u_\star}}\, C_T^2(x_o, z = l) \qquad (19a)$$

with

$$C_T^2(x_0, z = l) = \frac{\gamma \Theta_\star^2}{(l\kappa)^{\frac{2}{3}}} \left(0.75 + \frac{l}{L} 4.7\right). \qquad (19b)$$

These results are only valid in case $L_y = \infty$. The influence of finite $L_y$ is not taken into account. For finite $L_y$ the flow can go around the hill and the turbulence increases less than in the 2D case. This means that the temperature fluctuations do not get mixed so efficiently and it is expected that the seeing is higher than pedicted by 2D inner-layer theory.

## THE PU MODEL

The pressure that drives the flow over a hill of small slope can be approximated by the potential pressure (Prandtl's boundary layer). The correct way of proceeding would be to write the flow equations in a boundary following coordinate system. The assumption that the pressure is potential is only valid for small slopes and we will make a next assumption, also based on this fact. We stretch the surface of the hill into a straight line and assume that we can keep the Cartesian flow equations in this coordinate system while keeping the pressure as described by the potential pressure. We therefore assume that no transformation of the flow equations is necessary or in other words that the extra terms due to the curvature of the coordinate system are small.

From the equation of motion for potential flow we can easily derive an expression for the horizontal Fourier transform of the pressure $\hat{P}(\vec{k}, z)$.

$$\hat{P}(\vec{k}, z) = -\frac{(\vec{u}_0 \cdot \vec{k})^2}{|\vec{k}|} \hat{h} \exp(-|\vec{k}|\, z) \qquad (20)$$

The procedure is as follows. We specify the height of the terrain $h(x, y)$ and Fourier transform this into $\hat{h}(\vec{k})$. Substitution and a back Fourier transformation give $P(x, y, z)$. The value of the background wind $u_0(z)$ will be taken at a height $z = c\, |\vec{k}|^{-1}$, as used in Troen and de Baas (1987). The constant $c$ gives a tuning freedom. With this the pressure is determined and kept fixed in the further proceedings.

# RESULTS

## THE PU-MODEL APPLIED TO 2D FLOW OVER HILLS

First the constant in the potential pressure formulation was tuned against the Askervein measurements. The vertical wind profile at the hill top and the wind at 10 m height across the hill were checked against the measurements. A good accordance was found for $c = 1$.

Then the model could be tested against the 2D inner layer theory. These equations were confirmed and it was indeed found that the inner layer height $\ell$ is independent of the height of the hill and the maximum speedup is independent of the roughness.

The PU-model was checked against the second-order model of Zeman and Jensen (1987) which also accounts for curvature effects. Both models use a potential pressure. The results for a low slope Lorenzian hill

$$\eta = \frac{h}{1 + (x/L)^2}, \qquad (21)$$

were compared with $h = 50$ m, $L = 250$ m and $z_o = 0.03$ m. Speedup, kinetic energy and dissipation profiles before, at and after the hill top were compared (Fig. 1). We can see that the second order model gives more difference in the speedup as function of downwind distance. The profiles, also the ones for kinetic energy and dissipation, differ most behind the hill. However, considering the much larger sophistication of the Zeman and Jensen model the overall agreement is much better than expected.

We conclude that the E-$\varepsilon$ turbulence model is very useful and that the violation of the flow equations without making the correct boundary following coordinate transformation is of minor importance.

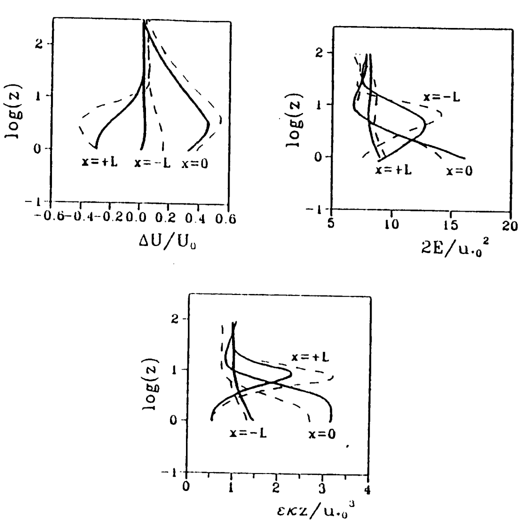


Fig. 1. Comparison of the E-$\varepsilon$ model (solid lines) and the second order model by Zeman and Jensen (dashed lines) for a Lorenzian hill with $h$=50 m, L = 250 m, and $z_0$ = 0.03 m a) speedup, b) kinetic energy, c) dissipation.

THE 'FLOW' AND 'TEMPERATURE' EQUATIONS APPLIED TO FLOW OVER HILLS

Characteristic for a nonneutral boundary layer is that the heat flux at the ground is non-zero, while zero at the top. In a homogeneous terrain this means that the diffusive term in the temperature equation is non-zero and the solution is necessarily nonstationary. However, to reduce the output of our model we restrict ourselves to stationary situations and are thus forced to consider only constant flux regions (surface layers). These regions extend to heights that are luckily sufficient for our purpose of predicting the seeing in the first 100m.

With the analytical inlet profiles of $\overline{\theta^2}$, $\varepsilon_\theta$, the $U, E, \varepsilon$ fields calculated with the PU model and the analytical $T$ field throughout the domain, we solve the equations for $\overline{\theta^2}$ and $\varepsilon_\theta$ and derive $C_T^2$. The equation for $\varepsilon_\theta$ contains three constants with a wide variety possible. This gave us the opportunity to tune the constants to the measurements made at La Silla (VLT report no 55, p88). In our model La Silla is described by a Lorenzian hill with $h$=300 m, $L_x$=1 km, $L_y$=100 km, $z_0$=.01 m, and the measurement conditions by $u_*$=.6 m/s and $\theta = 0.09$K.

The different set of constants tried are :

set 1 : $\{a_1, a_2, a_3\}$ ={1,1,1},
set 2 : $\{a_1, a_2, a_3\}$ ={2.36, 2.02, 1.5} and
set 3 : $\{a_1, a_2, a_3\}$ ={5, 3.38, 3.38}.

The different $C_T^2$ profiles at the top of the hill for set 1, 2 and 3 are shown in Fig. 2.
Only set 1 gives the undisturbed flow at higher heights in this case of stable flow over a hill (not shown here). The set $\{a_1, a_2, a_3\}$ ={1,1,1} represents the data best if the constant $\gamma$ in the $C_T^2$ relation to $\varepsilon$ and $\varepsilon_\theta$ (Eq(8)) is set equal to $\gamma$=1.6. Set 2 would require a larger $\gamma$ causing unrealistic values for $C_t^2$ at higher heights. The reverse is true for set 3. Here smaller values are required whereas the theory discussed before indicates that $\gamma$ should rather be larger.

We settle for the combination of set 1 and $\gamma$=1.6.

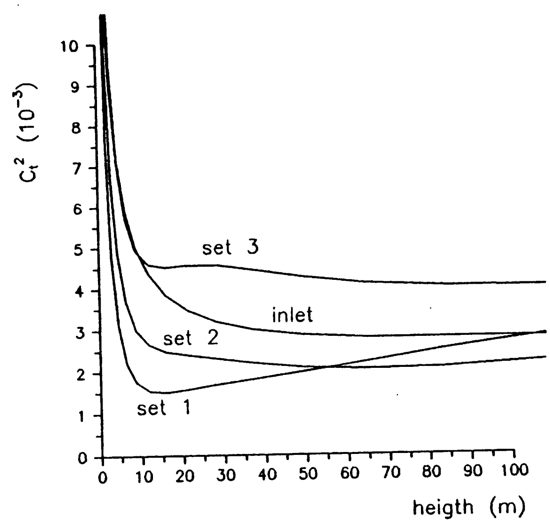


Fig. 2. Measurements (•) and calculated profile of the temperature structure function $C_t^2(z)$ (in $10^{-3}\,K^2\,m^{-2/3}$) on top of La Silla modelled by a Lorenzian hill with $h$=300 m, $L_x$=1000 m, $L_y$=100000 m, $z_0$=.01 m, $u_*$=.58 m/s for different constant sets $\{a_1, a_2, a_3\}$ in the $\varepsilon_\theta$ equation:
set 1 : $\{a_1, a_2, a_3\}$ ={1,1,1},
set 2 : $\{a_1, a_2, a_3\}$ ={2.36, 2.02, 1.5} and
set 3 : $\{a_1, a_2, a_3\}$ ={5, 3.38, 3.38}.
◇ inner layer results.

## MODELLING OF LA SILLA, PARANAL, MONTURA AND ARMAZONI

Different mountains in Chile were considered as potential sites for the VLT and measuring campaigns were set up. A first preliminary modeling of the mountains Paranal, Armazoni and Montura is made with Lorenzian shapes. The combinations of variables determining the hill shape are chosen so that La Silla, Paranal, Montura and Armazoni were approximated as best we could.

| mountain | h | $L_x$ | $L_y$ | $z_o$ | $\ell$ | $\Delta C_T^2$ | $C_T^2(\ell)$ |
|---|---|---|---|---|---|---|---|
| La Silla | 300 m | 1 km | 100 km | .01 m | 6 m | .62 | 3.84 |
| Paranal | 600 m | 1 km | 1 km | .1 m | 14 m | .45 | 1.59 |
| Montura | 300 m | 1 km | .5 km | .05 m | 11 m | .62 | 2.59 |
| Armazoni | 300 m | 1 km | 1 km | .01 m | 6 m | .62 | 3.84 |

Measurements are found in VLT report No. 55 and VLT report No. 62, p. 106, Fig. 4.72. The model results fit the data very well (Fig. 3).

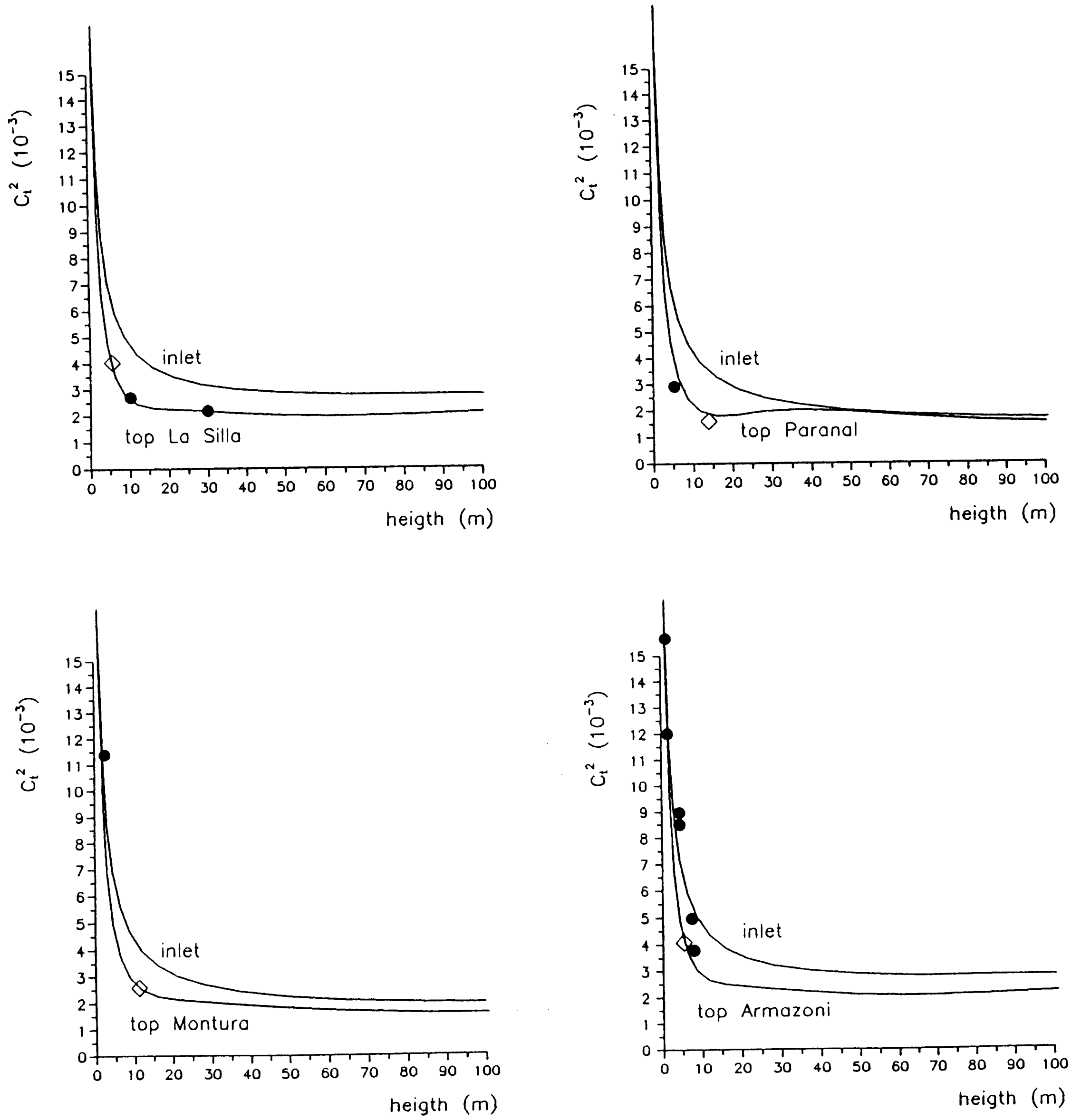


Fig. 3. Measurements (•) and calculated profile of the temperature structure function $C_t^2(z)$ (in $10^{-3}\ K^2 m^{-2/3}$)
a) on top of La Silla modelled by a Lorenzian hill with $h, L_x, L_y, z_0$ = 300, 1000, 100000, .01 m and $u_\star$ =.58 m/s
b) on top of Paranal modelled by a Lorenzian hill with $h, L_x, L_y, z_0$ = 600, 1000, 1000, .1 m and $u_\star$ =.87 m/s
c) on top of Montura modelled by a Lorenzian hill with $h, L_x, L_y, z_0$ = 300, 1000, 500, .05 m and $u_\star$ =.75 m/s
d) on top of Armazoni modelled by a Lorenzian hill with $h, L_x, L_y, z_0$ = 300, 1000, 1000, .01 m and $u_\star$ =.58 m/s
◇ inner layer results.

## DISCUSSION

We feel that our model has the following shortcomings:

- The assumption in the PU model that the pressure is potential and that the extra terms are small in the equation of motion due to the curvature of the coordinate system is only valid for flow over small slopes. This means that we have to restrict ourselves to lower hills.
- The vertical velocity plays a very important role in the temperature equation. A velocity of 3 cm/s already has an effect of a°C. In our stretched coordinate model we were forced to exclude the temperature from calculation. For the cases considered in this report that was not of great importance as the thermographic investigations of the undeveloped sites showed that the hills were thermally homogeneous (VLT report No.62, page 142). However in order to catch influences of heat flux differences occuring due to roads and buildings we would need to calculate the temperature distribution in the flow.

To be able to improve on all these shortcomings a new model is being developed at Risø National Laboratory. The flow equations are written in a boundary following coordinate system, enabling us to calculate separation in front and behind obstructions as well as the temperature distribution in the air.

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